A Generalization of the Symmetric Translog Functional Form

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Abstract

Feenstra and Weinstein introduced a special case of the translog cost function, the Symmetric Translog (ST) Cost Function, which replaced all of the second order parameters which are present in the Translog functional form by a single positive parameter. The ST functional form is useful in the context of estimating consumer preferences over related products and provides an alternative to the use of Constant Elasticity of Substitution (CES) preferences in this context. The ST functional form has a potential advantage over the CES functional form that it produces finite estimates for reservation prices for missing products whereas the CES reservation prices are always infinite. The present paper proposes a Generalized Symmetric Translog Cost Function which also generates finite reservation prices but is more flexible than the ST functional form.

JEL Classification Numbers

C22, D12

Key Words

Translog functional form, Symmetric Translog functional form, Generalized Symmetric Translog functional form, consumer demand systems, reservation prices, elasticities of substitution.

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1. Introduction

Detailed price and quantity data on purchases of consumer goods by households or similar data on sales of consumer goods by retailers is increasingly available to national statistical agencies and to academic researchers. An important characteristic of these data sets is the fact that, typically, there are many missing observations. Or put another way, there is tremendous product churn in these data sets. This causes problems both for statistical agencies constructing Consumer Price Indexes and for economists who fit systems of consumer demand functions based on utility maximizing behavior in order to measure welfare change for households.

Index number theory relies heavily on a matched model methodology: the price of a product in the current period is compared with the price of the identical product in a previous period. These price ratios that compare the price of a product over two periods are then aggregated up according to their economic importance to generate an overall estimate of consumer price inflation for the product category under consideration. This methodology fails when there are missing prices. Statistical agencies attempt to deal with the missing prices problem by either: (i) carrying forward the last available price and using this price as an imputation for a missing current price or (ii) using the movements of product prices that are available in both periods under consideration to impute the movement of a missing price. The first strategy is obviously flawed if there is significant general inflation or deflation in the product category under consideration. The second strategy is also suspect from the viewpoint of the economic approach to index number theory. When a product disappears, economic theory suggests that an appropriate imputed price for the disappearing product is a *reservation price*, which is the price which is just high enough to induce the consumer to buy zero units of the product.[2] Reservation prices will typically be higher than carry forward prices or product category inflation adjusted carry forward prices so using any form of carry forward price in place of a missing price leads to possible bias in official Consumer Price Indexes.

Standard consumer demand theory also fails when there are a large number of missing products. Standard consumer demand theory works as follows. A functional form for the utility function or dual expenditure function for a representative household is assumed along with the assumption of utility maximizing behavior. A system of demand equations is the result with quantities purchased as functions of the prices of the products and "income" (or more accurately, total expenditure on the group of products that is in scope). Then standard econometric software is used to estimate the unknown parameters in a nonlinear systems approach to the estimation. However, when there are missing prices, the missing prices have to be replaced with reservation prices which are unknown. Thus standard consumer demand theory fails in the missing prices context.

[2] The concept of a reservation price is due to Hicks (1940; 140).

There is an exception to the general failure of the standard approach to fitting systems of consumer demand functions. If the assumed functional form for the underlying utility function is a Constant Elasticity of Substitution (CES) utility function, then standard consumer demand theory can be adapted to deal with this particular functional form.[3] Furthermore, Feenstra (1994), in a path breaking paper, showed how standard index number theory could be adapted to deal with new and disappearing products in the CES context, provided one had an estimate of the (constant) elasticity of substitution that is a key parameter of the CES functional form.

However, a potential problem with Feenstra's CES methodology is that the implied reservation price for a missing product is infinite, which does not seem to be consistent with consumer behavior; i.e., we expect reservation prices to be finite. This is where the recent paper by Feenstra and Weinstein (2017) comes into play. They discovered a special case of the Translog functional form for a unit cost or expenditure function which can deal with new and disappearing products and it generates *finite reservation prices* for the missing products. Like the CES functional form, their *Symmetric Translog unit cost* function has enough free parameters to provide a first order approximation to an arbitrary unit cost function plus one addition free parameter to model substitution between the various products. The contribution of this paper is to provide a generalization of the Symmetric Translog functional form that has N free parameters to model substitution possibilities between the N products in scope instead of just one free parameter.

In section 2, we provide some background information on the general translog unit cost function. Section 3 introduces the Generalized Symmetric Translog (GST) unit cost function and derives the estimating equations generated by this functional form. Section 4 concludes.

2. Curvature Conditions for the Translog Functional Form

The Feenstra and Weinstein (2017) functional form for a unit cost function is a special case of the general Translog unit cost function proposed by Christensen, Jorgenson and Lau (1975). Our generalization of the Feenstra-Weinstein functional form is also a special case of the Translog unit cost function. Thus it will be useful to derive some of the properties of the Translog functional form in this section before introducing special cases of it.

Assume that purchasers of a group of N products have homothetic preferences that can be represented by a unit cost or expenditure function, $c(p)$, where $p \equiv [p_1,...,p_N]$ is a vector of positive consumer prices that purchasers face during a period. Denote the vector of quantities that are purchased in the period by $q \equiv [q_1,...,q_N]$.

[3] See Arrow, Chenery, Minhas and Solow (1961) for the CES functional form in the production function context and see Diewert (2020a) for materials on estimating the CES functional form in the consumer context.

Define the logarithm of the *translog unit expenditure function* c(p) as follows:[4]

(1) $\ln c(p) \equiv \alpha_0 + \Sigma_{n=1}^{N} \alpha_n \ln p_n + ½ \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} \alpha_{ij} \ln p_i \ln p_j$

where the parameters satisfy the restrictions $\Sigma_{n=1}^{N} \alpha_n = 1$; $\alpha_{ij} = \alpha_{ji}$ for all i, j and $\Sigma_{j=1}^{N} \alpha_{ij} = 0$ for i = 1,...,N. Differentiating the logarithm of the general translog unit expenditure function defined by (1) with respect to the logarithm of the ith price leads to the following first order logarithmic partial derivatives:

(2) $\partial \ln c(p)/\partial \ln p_i = \alpha_i + \Sigma_{j=1}^{N} \alpha_{ij} \ln p_j$ i = 1,...,N.

Suppose there are K utility maximizing purchasers of the N products in the period under consideration and they all have preferences that are dual to the Translog unit expenditure function c(p) defined by (1).[5] Let $q^k \equiv [q_1^k,\ldots,q_N^k]$ denote the utility maximizing quantity vector and u^k as the utility level for purchaser k for k = 1,…,K. We assume all purchasers face the same price vector p. Using Shephard's (1953; 11) Lemma, the demand vector for purchaser k is $q^k = \nabla_p c(p) u^k$ for k = 1,…,K where $\nabla_p c(p)$ is the vector of first order partial derivatives of the unit expenditure function, $[c_1(p),\ldots,c_N(p)]$, where $c_i(p) \equiv \partial c(p)/\partial p_i$ for i = 1,…,N. The *aggregate demand vector* q can be obtained by summing the q^k over the K purchasers:

(3) $q \equiv \Sigma_{k=1}^{K} q^k = \Sigma_{k=1}^{K} \nabla_p c(p) u^k = \nabla_p c(p)\, \Sigma_{k=1}^{K} u^k$.

Denote the inner product of the vectors p and q by $p \cdot q \equiv \Sigma_{n=1}^{N} p_i q_i$. *The ith expenditure share*, $s_i(p)$, regarded as a function of p is defined as follows:

(4) $s_i(p) \equiv p_i q_i(p)/p \cdot q(p)$ i = 1,…,N
$= p_i c_i(p)\, \Sigma_{k=1}^{K} u^k / p \cdot \nabla_p c(p)\, \Sigma_{k=1}^{K} u^k$ using (3)
$= p_i c_i(p)/c(p)$ [6]
$= \partial \ln c(p)/\partial \ln p_i$.

Let the period t price vector be p^t and let the period t market expenditure share vector be s^t. Then using equations (2) and (4), the following system of expenditure share equations can be derived:

(5) $s_i^t = \alpha_i + \Sigma_{j=1}^{N} \alpha_{ij} \ln p_j^t$; i = 1,...,N.

The above system of share equations can serve as estimating equations in order to determine the unknown parameters in the Translog unit expenditure function defined by (1), provided that there are no missing products and all product prices p_i are positive.

[4] See Christensen, Jorgenson and Lau (1975) and Diewert (1974) (1976) on the translog functional form in the consumer context.
[5] See Diewert (1974) for the details of this duality model.
[6] The restrictions on the c(p) defined by (1) imply that c(p) is linearly homogeneous in the components of p, Thus by Euler's Theorem on homogeneous functions, $c(p) = p \cdot \nabla c(p)$.

We need to check that the estimated Translog expenditure function satisfies the condition that the unit cost function be concave in prices. Thus we need to calculate the matrix of second order derivatives of the estimated unit cost function, c(p), and check whether this matrix is negative semidefinite at the sample prices p^t.

Combining equations (2) and (4) leads to equations (6) for an arbitrary p vector:

$$(6)\ s_i(p) = \alpha_i + \Sigma_{j=1}^{N} \alpha_{ij}\ln p_j = \partial \ln c(p)/\partial \ln p_i \qquad i = 1,...,N.$$

If prices and expenditures are positive, then the logarithmic derivatives $\partial \ln c(p)/\partial \ln p_i$ are equal to $[p_i/c(p)]\partial c(p)/\partial p_i$ for i = 1,...,N. Using these relationships and equations (6) for the share functions $s_i(p)$, we obtain the following expressions for the first order partial derivatives of the translog cost function c(p):

$$(7)\ c_i(p) \equiv \partial c(p)/\partial p_i = c(p)s_i(p)/p_i\ ; \qquad i = 1,...,N.$$

Now differentiate $c_i(p)$ defined by (7) with respect to p_j for $i \neq j$ in order to obtain the following expressions for the second order partial derivatives of c(p):

$$\begin{aligned}(8)\ c_{ij}(p) &\equiv \partial^2 c(p)/\partial p_i \partial p_j; && i = 1,...,N;\ j = 1,...N;\ i \neq j \\ &= c_j(p)s_i(p)p_i^{-1} + c(p)p_i^{-1}[\partial s_i(p)/\partial \ln p_j][\partial \ln p_j/\partial p_j] && \text{differentiating (7)} \\ &= [c(p)s_j(p)p_j^{-1}]s_i(p)p_i^{-1} + c(p)p_i^{-1}[\alpha_{ij}]p_j^{-1} \\ & && \text{using (7) for } i = j \text{ and differentiating } s_i(p) \\ & && \text{defined by (6) with respect to } \ln p_j \\ &= c(p)p_i^{-1}p_j^{-1}[\alpha_{ij} + s_i(p)s_j(p)].\end{aligned}$$

Using similar arguments, we obtain the following expressions for the second order partial derivatives of c(p) with respect to p_i and p_i again:

$$\begin{aligned}(9)\ c_{ii}(p) &\equiv \partial^2 c(p)/\partial p_i \partial p_i; && i = 1,...,N \\ &= c(p)p_i^{-1}p_j^{-1}[\alpha_{ii} + s_i(p)s_i(p) - s_i(p)].\end{aligned}$$

Make an N by N matrix where the ijth element is equal to $[\alpha_{ij} + s_i(p)s_j(p)]$ if $i \neq j$ and is equal to $[\alpha_{ii} + s_i(p)s_i(p) - s_i(p)]$ if i = j. This matrix is equal to $A + ss^T - \hat{s}$ where $A \equiv [\alpha_{ij}]$ is the N by N symmetric matrix of second order coefficients for the translog functional form defined by (1), $s \equiv [s_1(p),...,s_N(p)]^T$ is the column vector of shares defined by (6) and $\hat{s}$ is a diagonal N by N matrix that has the elements of s on the main diagonal. Looking at (8) and (9), and assuming that all prices are positive, it can be seen that necessary and sufficient conditions for the N by N matrix of second order derivatives of the unit cost function c(p), $\nabla^2 c(p)$, to be negative semidefinite are that $A + ss^T - \hat{s}$ be negative semidefinite.

It can be shown that the matrix $ss^T - \hat{s}$ is negative semidefinite provided that the share vector s is nonnegative.[7] Thus a sufficient condition for c(p) to satisfy the concavity conditions at a point p (where $p >> 0_N$ so that all components of the price vector p are positive) is that the A matrix be negative semidefinite since this will imply that $\nabla^2 c(p) = c(p)\,\hat{p}^{-1}[A + ss^T - \hat{s}\,]\,\hat{p}^{-1}$ is negative semidefinite.[8] In fact, if A is negative semidefinite, then c(p) defined by (1) will be concave over all positive prices p that generate nonnegative demand vectors.

The Allen (1938; 504) Uzawa (1962) *elasticity of substitution* between commodities i and j (where $i \neq j$), $\sigma_{ij}(p)$, is defined as follows:

$$(10)\ \sigma_{ij}(p) \equiv c(p)c_{ij}(p)/c_i(p)c_j(p)\,; \qquad i = 1,...,N;\ j = 1,...,N;\ i \neq j$$
$$= 1 + [\alpha_{ij}/s_i(p)s_j(p)] \qquad \text{using (7) and (8).}$$

Note that when all second order terms α_{ij} are equal to 0, the Translog unit expenditure function collapses down to the Cobb Douglas (1928) functional form and all elasticities of substitution are equal to unity, which is a well known result.

There are N(N−1)/2 independent α_{ij} parameters in the general case which determine the N(N+1)/2 independent elasticities of substitution σ_{ij}. As N increases, the number of independent parameters in the general Translog functional form increases to approximately $N^2/2$ which will be impossible to estimate accurately (or at all). Thus in the following section, we suggest a Translog model that has N free parameters to model substitution effects. This special case turns out to be a generalization of the Feenstra and Weinstein Symmetric Translog expenditure function.

3. A Generalization of the Symmetric Translog Expenditure Function

As in the previous section, let A represent the N by N symmetric matrix of second order parameters (the α_{ij}) in the translog unit cost function c(p) defined by (1). Let $\beta \equiv [\beta_1,...,\beta_N]^T$ be a column vector of nonnegative parameters and define the A matrix as follows:

$$(11)\ A \equiv \beta\beta^T - (\beta^T 1_N)\,\hat{\beta}$$

where 1_N is a column vector of ones of dimension N and $\hat{\beta}$ is a diagonal N by N matrix with the elements of the vector β running down the main diagonal of the matrix. Thus for $i \neq j$, $\alpha_{ij} \equiv \beta_i\beta_j$ and for i = j, $\alpha_{ii} \equiv \beta_i^2 - \beta_i(\Sigma_{n=1}^N \beta_n)$. It is straightforward to verify that $A1_N = 0_N$ where 0_N is a vector of 0's of dimension N. These restrictions must be satisfied in

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[7] We need to show that $[\Sigma_{n=1}^N x_n s_n]^2 - \Sigma_{n=1}^N x_n s_n x_n \leq 0$ for all $x_1,...,x_N$. $[\Sigma_{n=1}^N x_n s_n]^2 = [\Sigma_{n=1}^N (x_n s_n^{1/2})(s_n^{1/2})]^2 \leq [\Sigma_{n=1}^N (x_n s_n^{1/2})(x_n s_n^{1/2})][\Sigma_{n=1}^N (s_n^{1/2})(s_n^{1/2})]$ using the Cauchy Schwarz inequality $= [\Sigma_{n=1}^N x_n s_n x_n][\Sigma_{n=1}^N s_n] = [\Sigma_{n=1}^N x_n s_n x_n]$ since $\Sigma_{n=1}^N s_n = 1$.

[8] The N by N matrix $\hat{p}^{-1}$ is equal to the inverse of the N by N diagonal matrix that has the elements of the positive price vector p running down the main diagonal.

order for the translog unit cost function to be linearly homogeneous in the vector of prices p.

It is also possible to show that the A defined by (11) is a negative semidefinite symmetric matrix. The symmetry property is obvious. To show that A is negative semidefinite, we need to show that $x^TAx = (x^T\beta)^2 - (\Sigma_{n=1}^N \beta_n)(\Sigma_{n=1}^N x_n\beta_n x_n) \le 0$ for all vectors $x \equiv [x_1,...,x_N]^T$. We have the following equalities and inequalities:

$$
\begin{aligned}
(12)\ (x^T\beta)^2 &= (x^T\beta)^2 \\
&= [\Sigma_{n=1}^N (x_n\beta_n^{1/2})(\beta_n^{1/2})]^2 \\
&\le [\Sigma_{n=1}^N (x_n\beta_n^{1/2})(x_n\beta_n^{1/2})][\Sigma_{n=1}^N (\beta_n^{1/2})(\beta_n^{1/2})] \\
&\qquad\qquad \text{using the Cauchy-Schwarz inequality} \\
&= [\Sigma_{n=1}^N x_n\beta_n x_n][\Sigma_{n=1}^N \beta_n]
\end{aligned}
$$

which is equivalent to the desired inequality. Note that we require $\beta_n \ge 0$ so that the positive square root of each β_n is well defined.

Assume that we have data on prices and quantities purchased for the N products for T periods; denote the price and quantity of product n in period t by p_n^t and q_n^t respectively for t = 1,...,T and n = 1,...,N. We need to allow for the possibility of missing products in one or more periods.[9] Let S(t) and N(t) be the set and number of products that are sold in period t. If product n is not purchased in period t, then $q_n^t = 0$ and the corresponding price is a *reservation price* p_n^{t*} to be determined below. The expenditure share for product n that is sold in period t is $s_n^t \equiv p_n^t q_n^t/p^t\cdot q^t$ where $p^t\cdot q^t$ is the inner product of the period t price and quantity vectors. If product n is not sold in period t, then $s_n^t \equiv 0$.

Recall that the system of share equations for the general translog unit cost function was defined by equations (5) for the period t data. Below, we replace the general A matrix used in equations (5) with the A matrix defined by (11) above. We obtain the following system of share equations for period t:

$$
\begin{aligned}
(13)\ s_i^t &= \alpha_i + \Sigma_{j=1}^N \alpha_{ij}\ln p_j^t\ ; \qquad\qquad t = 1,...,T;\ i = 1,...,N \\
&= \alpha_i - \beta_i(\Sigma_{n=1}^N \beta_n)\ln p_i^t + \beta_i(\Sigma_{n=1}^N \beta_n \ln p_n^t) \\
&= \alpha_i - \beta_i(\Sigma_{n=1}^N \beta_n)\ln p_i^t + \beta_i(\Sigma_{n\in S(t)} \beta_n \ln p_n^t) + \beta_i(\Sigma_{n\notin S(t)} \beta_n \ln p_n^t)
\end{aligned}
$$

If there are no missing products in period t, then the second line in (13) can be used as a basis for estimating equations; all period t log prices $\ln p_i^t$ can be calculated. However, if there are missing products in period t, then the third line in (13) shows us that there is a problem: the prices p_n^t for the missing products n that do not belong to S(t) cannot be

[9] In applications of this functional form to scanner data from retailers, what is typically available are data on products sold during a period along with their unit value price. Thus a product could be available during a period but if it is not sold, it will be regarded as a missing product for our purposes.

observed.[10] Thus we cannot calculate the terms $\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t$. Below, we will use the same method that was used by Feenstra and Weinstein (2017) to address this problem.[11]

If there are missing products in period t, then sum the shares s_i^t for $i\in S(t)$; i.e., form the sum of the shares for products i that are actually purchased in period t. This purchased share sum will equal 1. Thus we obtain the following equation:

$$
\begin{aligned}
(14)\ 1 &= \Sigma_{i\in S(t)}\, \alpha_i - (\Sigma_{n=1}^{N}\, \beta_n)(\Sigma_{i\in S(t)}\, \beta_i \ln p_i^t) + (\Sigma_{i\in S(t)}\, \beta_i)(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t) \\
&\qquad + (\Sigma_{i\in S(t)}\, \beta_i)(\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t) \\
&= \Sigma_{n\in S(t)}\, \alpha_n - (\Sigma_{n\notin S(t)}\, \beta_n)(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t) + (\Sigma_{i\in S(t)}\, \beta_i)(\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t)
\end{aligned}
$$

Since $1 = \Sigma_{n=1}^{N}\, \alpha_n = \Sigma_{n\in S(t)}\, \alpha_n + \Sigma_{n\notin S(t)}\, \alpha_n$, we see that $1 - \Sigma_{n\in S(t)}\, \alpha_n = \Sigma_{n\notin S(t)}\, \alpha_n$. This equation and (14) imply the following expression for the sum of the terms involving the unobserved reservation prices, $\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t$:

$$
(15)\ \Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t = (\Sigma_{i\in S(t)}\, \beta_i)^{-1}(\Sigma_{n\notin S(t)}\, \alpha_n) + (\Sigma_{i\in S(t)}\, \beta_i)^{-1}(\Sigma_{n\notin S(t)}\, \beta_n)(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t).
$$
[12]

Now use the right hand side of equation (15) to replace the term involving reservation prices in (13), $\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t$, for all products i that are purchased in period t. We obtain the following system of estimating equations (for products i that are available) if there are missing products in period t:

$$
\begin{aligned}
(16)\ s_i^t &= \alpha_i - \beta_i(\Sigma_{n=1}^{N}\, \beta_n)\ln p_i^t + \beta_i(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t) + \beta_i(\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t)\ ; \qquad i\in S(t) \\
&= \alpha_i - \beta_i(\Sigma_{n=1}^{N}\, \beta_n)\ln p_i^t + \beta_i(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t) + \beta_i(\Sigma_{n\in S(t)}\, \beta_n)^{-1}(\Sigma_{n\notin S(t)}\, \alpha_n) \\
&\qquad + \beta_i(\Sigma_{n\in S(t)}\, \beta_n)^{-1}(\Sigma_{n\notin S(t)}\, \beta_n)(\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t) \\
&= \alpha_i + \beta_i(\Sigma_{n\in S(t)}\, \beta_n)^{-1}(\Sigma_{n\notin S(t)}\, \alpha_n) - \beta_i(\Sigma_{n=1}^{N}\, \beta_n)[\ln p_i^t - \ln p_\beta^t]\ .
\end{aligned}
$$

The logarithm of the *period t beta weighted average price* p_β^t for products that are available in period t is defined as follows (if there are missing products in period t):

$$
(17)\ \ln p_\beta^t \equiv (\Sigma_{n\in S(t)}\, \beta_n \ln p_n^t)/(\Sigma_{n\in S(t)}\, \beta_n).
$$

If there are no missing products in period t, then use the second line in equations (13) as the estimating equation. If there are missing products in period t, then use the second line in (16) as the estimating equation. The restriction $\Sigma_{i=1}^{N}\, \alpha_i = 1$ must be imposed on the estimating equations. It can be seen that if there are missing products, then summing equations on both sides of equations (16) over all $i\in S(t)$ leads to the equation $1 = 1$.

If there are missing products and the estimated β_i turns out to be positive, it can be seen the first two terms on the right hand side of the last equation in (16) are constants and the

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[10] For missing products n in period t, the appropriate prices to use in equations (24) for these missing products are reservation prices: prices p_n^{t*} which are just high enough to induce consumers to purchase 0 units of these products.

[11] Feenstra (1994) used a similar technique in the CES context when there are missing products.

[12] If there are no missing prices in period t, then define $\Sigma_{n\notin S(t)}\, \beta_n \ln p_n^t = \Sigma_{n\notin S(t)}\, \alpha_n = \Sigma_{n\notin S(t)}\, \beta_n \equiv 0$.

third term, $-\beta_i(\Sigma_{n=1}^N \beta_n)[\ln p_i^t - \ln p_\beta^t]$, has a straightforward interpretation. The period t price p_β^t defined by (17) is a beta weighted geometric average of the prices of the products that are available in period t. If all of the β_n are positive, the last line in (16) shows that if the price of product i in period t, p_i^t, increases, then the period t expenditure share for that product will decrease. The constant term α_i can be interpreted as the constant expenditure share for product i if preferences are Cobb Douglas. In the Cobb Douglas case, all of the β_n are equal to 0. But Cobb Douglas preferences are not very sensible when there are missing products because Cobb Douglas preferences imply constant expenditure shares, which cannot be the case if products vary in their availability across time periods. The Generalized Symmetric Translog functional form makes an adjustment to the Cobb Douglas constant share term α_i by adding the term $\beta_i(\Sigma_{n\in S(t)} \beta_n)^{-1}(\Sigma_{n\notin S(t)} \alpha_n)$ to it. Note that this adjustment term varies over periods if product availability varies over periods. Assuming that the α_n are all positive, the more missing products that there are in period t, the bigger will be this addition to α_i.[13]

Note that nonnegativity of the α_n and β_n parameters can be imposed on the nonlinear regression model that is defined by equations (13) (if there are no missing products in period t) and the second line in (16) (if there are missing products in period t) by squaring the parameters.

Once estimates for the α_n and β_n are available and there are missing products, then the logarithms of the *reservation prices* p_i^{t*} can be calculated by setting $s_i^t = 0$ in equation (16) (provided $\beta_i > 0$) and solving the resulting equation for $\ln p_i^{t*}$ as follows:

$$(18)\ \ln p_i^{t*} = \ln p_\beta^t + [\alpha_i/\beta_i(\Sigma_{n=1}^N \beta_n)] + [(\Sigma_{n\notin S(t)} \alpha_n)/(\Sigma_{n\in S(t)} \beta_n)(\Sigma_{n=1}^N \beta_n)]\ ; \qquad i\notin S(t).$$

It turns out that $\ln p_i^{t*}$ (and p_i^{t*}) will be infinite if β_i equals zero.[14]

Finally, we specialize the general Translog formula (10) for the elasticity of substitution between products i and to the case of the Generalized Symmetric Translog. Using assumptions (11) on the parameters of the A matrix, the elasticity of substitution between products i and j in period t for our Generalized Symmetric Translog preferences is given by:

$$(19)\ \sigma_{ij}(p^t) = 1 + [\beta_i\beta_j/s_i(p^t)s_j(p^t)]\ ; \qquad i \neq j.$$

Thus if β_i is large, then product i will tend to have a high elasticity of substitution with all other products that are purchased in period t. Note that formula (19) requires that the

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[13] For global monotonicity of the Generalized Symmetric Translog functional form, we require that each α_i be nonnegative. However, for local monotonicity, it is not necessary that all α_n be nonnegative. A negative α_i can be offset by a sufficiently large and positive β_i coefficient.

[14] In the case where all of the estimated β_i turn out to be positive, then all of the reservation prices generated by this model will be finite. Note that reservation prices for missing products in the CES model are infinite; see Feenstra (1994).

fitted shares, $s_i(p^t)$ and $s_j(p^t)$, be positive. This will always be the case if the estimated α_n and β_n are all positive.

Note that formula (19) tells us something important: using this model, *the elasticity of substitution between any pair of products will be equal to or greater than one*. Thus this model should only be applied to a group of products that are quite substitutable with each other. It should not be applied to a group of products that are quite different since it is likely that some of the true elasticities of substitution between very different product groups will be less than one.

We now show that our Generalized Symmetric Translog model is a generalization of the Symmetric Translog (ST) model introduced by Feenstra and Weinstein (2017). The A matrix that corresponds to the Feenstra Weinstein model has the following form:

$$(20)\ A = -\gamma I_N + \gamma(1/N)1_N 1_N^T$$

where γ is a scalar parameter greater than 0, I_N is the N by N identity matrix and 1_N is a column vector of ones of dimension N. Recall from (11) that the A matrix for the Generalized Symmetric Translog unit cost function had the form $A = \beta\beta^T - (\beta^T 1_N)\hat{\beta}$. Set $\beta = (\gamma/N)^{1/2}1_N$ and substitute this value for β into (11). It can be verified that the resulting A matrix is equal to the right hand side of (20). Thus the Feenstra-Weinstein functional form is a special case of our GST functional form.

The elasticities of substitution for the Feenstra-Weinstein model are given by:

$$(21)\ \sigma_{ij}(p^t) = 1 + [\gamma/N s_i(p^t)s_j(p^t)] ; \qquad i \neq j.$$

If $\gamma > 0$, then all of the elasticities of substitution for the Feentra-Weinstein model will be greater than one and all reservation prices for missing products will be finite.

Both the Symmetric Translog and Generalized Symmetric Translog functional forms have the property that elasticities of substitution between any pair of products must be equal to or greater than one. This means that these functional forms should only be used when the products in scope are strong substitutes for each other.[15]

4. Conclusion

Some tentative conclusions are as follows:

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[15] Products i and j are strong substitutes if $\sigma_{ij} \geq 1$ and weak substitutes if $0 < \sigma_{ij} < 1$. Diewert (2020b) noted the importance of strong and weak substitution in deriving inequalities between various index number formulae. Both the CES and Feenstra Weinstein functional forms rule out the possibility of complementarity between any pair of products. Products i and j are complements if $\sigma_{ij} < 0$. See Hicks (1946; 311-312) on the definition of complementarity. The CES, Symmetric Translog and Generalized Symmetric Translog functional forms all rule out complementarity.

- The Feenstra Weinstein Symmetric Translog functional form offers a reasonable alternative to the use of the CES functional form. Both functional forms have only one free parameter to approximate $N(N-1)/2$ elasticities of substitution between all possible pairs of products.
- The CES functional form sets all possible elasticities of substitution $\sigma_{ij}(p)$ for $i \neq j$ equal to a constant $\sigma \geq 1$ while the Feenstra-Weinstein (FW) functional form sets $\sigma_{ij}(p) = 1 + [\gamma/Ns_i(p)s_j(p)]$ where γ is a positive parameter.
- The CES functional form has the possible disadvantage of setting all reservation prices for missing products equal to plus infinity whereas the ST functional form generates finite reservation prices provided the estimated γ parameter is positive.
- Our Generalized Symmetric Translog functional form generates finite reservation prices provided the estimated β_n parameters are positive. The estimated elasticities of substitution for period t are given by $\sigma_{ij}(p^t) = 1 +[\beta_i\beta_j/s_i(p^t)s_j(p^t)]$ provided that the fitted period t expenditure shares $s_i(p^t)$ and $s_j(p^t)$ are positive. This functional form allows for much more flexibility in the resulting elasticities of substitution.
- The disadvantage of the GST functional form is the complexity of the estimating equations. If all products are present in period t, the estimating equations are given by the second line in equations (13). The degree of nonlinearity in the coefficients of the log prices is not too bad in these equations. However, if there are missing products in period t, the estimating equations are given by the second line in equations (16). The nonlinearity in the coefficients of the log prices is much more severe in these equations.
- If some estimated β_i's turn out to be 0, then we can set those β_i equal to a small positive number and rerun the regression. Some preliminary results indicate that the resulting loss in log likelihood is small and the effect on the regression is minimal. However, the resulting finite reservation prices are determined by the arbitrary small number.[16]
- Both the ST and GST functional form models should only be applied to products which are highly substitutable with each other since the estimated elasticities of substitution will be equal to or greater than one for both models. .

It should be noted that the Generalized Symmetric Translog functional form is far from being a flexible functional form. However, in the context where the number of products in scope is large, it is impossible to estimate a fully flexible functional form: with N products, the number of parameters required to be a fully flexible unit cost function is $(N+1)N/2$ which is 5050 if $N = 100$. However, it is possible to estimate *semiflexible* functional forms.[17] In the present context with new and disappearing products, a way forward is to estimate a KBF utility function where curvature conditions can be imposed

[16] A referee suggested an alternative method for dealing with 0 estimates for some of the β_n: group some of the products into groups where a common set of β_n parameters are estimated.
[17] See Diewert and Wales (1988) on the concept of a semiflexible functional form.

and the overall number of parameters can also be controlled; see Diewert and Feenstra (2019) (2022) for examples applying this approach.[18]

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[18] For additional materials on how to impose curvature conditions on the KBF functional form, see Diewert (2018).

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