CONSUMER PRICE INDEX THEORY

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CHAPTER 3: THE AXIOMATIC OR TEST APPROACH TO INDEX NUMBER THEORY[1]

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1. Introduction

As was seen in Chapter 2, it was useful to be able to evaluate various index number formulae that have been proposed in terms of their properties. If a formula turns out to have rather undesirable properties, then doubt is cast on its suitability as an index that could be used by a statistical agency as a target index. Looking at the mathematical properties of index number formulae leads to the *test* or *axiomatic approach to index number theory*. In this approach, desirable properties for an index number formula are proposed and then it is attempted to determine whether any formula is consistent with these properties or tests. An ideal outcome is the situation where the proposed tests are both desirable and completely determine the functional form for the formula.

The axiomatic approach to index number theory is not completely straightforward, since choices have to be made in two dimensions:

- The index number framework must be determined.
- Once the framework has been decided upon, it must be decided what tests or properties should be imposed on the index number.

The second point is straightforward: different price statisticians may have different ideas about what tests are important and alternative sets of axioms can lead to alternative "best" index number functional forms. This point must be kept in mind while reading this chapter, since there is not universal agreement on what the "best" set of "reasonable" axioms is. Hence the axiomatic approach can lead to more than one "best" index number formula.

The first point about choices listed above requires further discussion. In the previous chapter, for the most part, the focus was on *bilateral index number theory*; i.e., it was assumed that prices and quantities for the same N commodities were given for two periods and the object of the index number formula was to compare the overall level of prices in one period with the other period. In this framework, both sets of price and quantity vectors were regarded as variables that could be *independently varied* so that for example, variations in the prices of one period did not affect the prices of the other period or the quantities in either period. The emphasis was on comparing the overall cost of a fixed basket of quantities in the two periods or taking averages of such fixed basket indexes. This is an example of an index number framework.

However, other index number frameworks are possible. For example, instead of decomposing a value ratio into a term that represents price change between the two periods times another term that represents quantity change, one could attempt to decompose a value aggregate for one period into a single number that represents the price level in the period times another number that represents the quantity level in the period. In this approach, the price level is supposed to be a function of the N commodity prices pertaining to that aggregate in the period under consideration and the quantity level is supposed to be a function of the N commodity quantities pertaining to the aggregate in the period. The resulting price level function was called an *absolute index number* by Frisch (1930; 397), a *price level* by Eichhorn (1978; 141) and a *unilateral price index* by Anderson, Jones and Nesmith (1997; 75). This approach to index number theory (in the context of the axiomatic approach) will be considered in section 2.[2]

[2] In Chapters 7 and 8, the levels approach will be considered again but from the viewpoint of the economic approach to index number theory rather than the test approach. Using the economic approach, quantities cannot be varied independently of prices.

The remaining approaches in this Chapter are largely bilateral approaches; i.e., the prices and quantities in an aggregate are compared for two periods. In sections 3 and 4 below, the value ratio decomposition approach is taken. In sections 3-6, the bilateral price and quantity indexes, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, are regarded as functions of the price vectors pertaining to the two periods, p^0 and p^1, and the two quantity vectors, q^0 and q^1. The axioms or tests that are placed on the price index function $P(p^0,p^1,q^0,q^1)$ not only reflect "reasonable" price index properties but some tests have their origin as "reasonable" tests on the companion quantity index $Q(p^0,p^1,q^0,q^1)$. The approach taken in sections 3 and 4 simultaneously determines the "best" price and quantity indexes. Section 5 looks at the test performance of various bilateral index number formulae.

The implications of the circularity test are examined in section 6. It turns out that only a few index number formulae satisfy this test. Recall that in Chapter 2, the Lowe index was introduced. This index does not fit precisely into the bilateral framework, since the quantity weights used in this index do not necessarily correspond to the quantities that pertain to either of the periods which are characterized by the price vectors p^0 and p^1. In section 6, the axiomatic properties of the class of indexes of the form $P(p^0,p^1,q)$ will be studied.

One of Walsh's (1921a) approaches to index number theory[3] was an attempt to determine the "best" weighted average of the price relatives, r_n, where $r_n \equiv p_n^1/p_n^0$ for $n = 1,...,N$. This is equivalent to using an axiomatic approach to try and determine the "best" index of the form $P^*(r,v^0,v^1)$ where $r \equiv [r_1,...,r_N]$ and v^t is a vector of expenditures on the N commodities during period t for $t = 0,1$. This approach will be considered in sections 7 and 8 below.[4]

In section 9, the index number framework explained in sections 7 and 8 is used to develop a methodological approach to the problem of decomposing overall price change into additive components that are functions of the percentage changes in individual commodity prices.

An Appendix provides proofs of some complex results.

2. The Test Approach to Index Number Theory using Price Levels

Denote the price and quantity of commodity i in period t by p_i^t and q_i^t respectively for $i = 1,2,\ldots,N$ and $t = 0,1,\ldots,T$. The variable q_i^t is interpreted as the total amount of commodity i transacted within period t. In order to conserve the value of transactions, it is necessary that p_i^t be defined as a unit value; i.e., p_i^t must be equal to the value of transactions in commodity i for period t divided by the total quantity transacted, q_i^t. In principle, the period of time should be chosen so that variations in commodity prices within a period are very small compared to their variations between periods.[5] For $t = 0,1,\ldots,T$, and $i = 1,\ldots,N$, define the value of transactions in commodity i as $v_i^t \equiv p_i^t q_i^t$ and define the *total value of transactions in period t* as:

[3] Walsh (1901) also considered basket type approaches to index number theory as was seen in Chapter 2.

[4] In section 7, rather than starting with indexes of the form $P(r,v^0,v^1)$, indexes of the form $P(p^0,p^1,v^0,v^1)$ are considered. However, if the invariance to changes in the units of measurement test is imposed on this index, it is equivalent to studying indexes of the form $P(r,v^0,v^1)$. Vartia (1976) also used a variation of this approach to index number theory.

[5] This treatment of prices as unit values over time follows Walsh (1901; 96) (1921a; 88) and Fisher (1922; 318). Fisher and Hicks both had the idea that the length of the period should be short enough so that variations in price for a single commodity within the period could be ignored as the following quotations indicate: "Throughout this book 'the price' of any commodity or 'the quantity' of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since,

(1) $V^t \equiv \Sigma_{i=1}^N v_i^t = \Sigma_{i=1}^N p_i^t q_i^t$; $t = 0,1,...,T.$

Using the above notation, the following *levels version of the index number problem* is defined as follows: for t = 0,1,…,T, find scalar numbers P^t and Q^t such that

(2) $V^t = P^t Q^t$; $t = 0,1,...,T.$

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; hence:

(3) $P^t = c(p^t)$; $Q^t = f(q^t)$; $t = 0,1,...,T.$

The functions c and f are to be determined somehow. Note that (3) requires that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This is a reasonable requirement since there is no reason to change the method of aggregation as time changes.

Substituting (3) and (2) into (1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

(4) $c(p)f(q) = \Sigma_{i=1}^N p_i q_i$ for all $p_i > 0$ and all $q_i > 0$.

It is natural to assume that the functions c(p) and f(q) be positive if all prices and quantities are positive:

(5) $c(p_1,...,p_N) > 0$; $f(q_1,...,q_N) > 0$ if all $p_i > 0$ and all $q_i > 0$.

Let 1_N denote an n dimensional vector of ones. Then (5) implies that when $p = 1_N$, $c(1_N)$ is a positive number, $a > 0$, and when $q = 1_N$, then $f(1_N)$ is also a positive number, $b > 0$; i.e., (5) implies that c and f satisfy:

(6) $c(1_N) = a > 0$; $f(1_N) = b > 0$.

Let $p = 1_N$ and substitute the first equation in (6) into (4) in order to obtain the following equation:

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ordinarily, the variation during a year, so far, at least, as prices are concerned, is too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold." Irving Fisher (1922; 318). "I shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets." John Hicks (1946; 122).

(7) $f(q) = \Sigma_{i=1}^{N} q_i/a$ for all $q_i > 0$.

Now let $q = 1_N$ and substitute the second equation in (6) into (4) in order to obtain the following equation:

(8) $c(p) = \Sigma_{i=1}^{N} p_i/b$ for all $p_i > 0$.

Finally substitute (7) and (8) into the left hand side of (4) and the following equation is obtained:

(9) $[\Sigma_{i=1}^{N} q_i/a][\Sigma_{i=1}^{N} p_i/b] = \Sigma_{i=1}^{N} p_iq_i$ for all $p_i > 0$ and all $q_i > 0$.

If N is greater than one, it is obvious that equation (9) cannot be satisfied for all strictly positive p and q vectors. Thus if the number of commodities N exceeds one, then there do not exist any functions c and f that satisfy (4) and (5).[6]

Thus this *levels test approach* to index number theory comes to an abrupt halt; it is fruitless to look for price and quantity level functions, $P^t = c(p^t)$ and $Q^t = f(q^t)$, that satisfy (2) or (4) and also satisfy the very reasonable positivity requirements (5).[7] Thus in the following sections of this chapter, the levels approach to price measurement will be replaced by the bilateral comparisons approach that was used in Chapter 2.

3. Tests for Bilateral Price Indexes

In this section and the following section, the strategy will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[8] The word "bilateral"[9] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, p^0,p^1,q^0,q^1, that are to be aggregated into a single number that summarizes the overall change in the N price ratios, $p_1^1/p_1^0,\ldots, p_N^1/p_N^0$.

The bilateral price index function, $P(p^0,p^1,q^0,q^1)$, is assumed to be well defined if all prices and quantities are positive for the two periods under consideration. If a commodity is missing in both periods, then it can simply be ignored. However, if it is missing in one period but not in the other period, this can create problems; i.e., $P(p^0,p^1,q^0,q^1)$ may not be well defined if one or more prices or quantities is equal to zero.[10] In this case where some prices and quantities may be equal to 0, we assume that the following conditions hold:[11]

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[6] Eichhorn (1978; 144) established this result.
[7] It is important to keep in mind that this result follows under the assumption that prices and quantities can vary *independently* from each other. When taking the economic approach to index number theory in Chapters 5 and 8, prices can vary independently but quantities will depend on prices so *quantities cannot vary independently from prices*. Thus when taking the economic approach, it is quite possible to find functions c(p) and f(q) such that $c(p)f(q) = \Sigma_{n=1}^{N} p_nq_n$.
[8] Much of the material in this section is drawn from sections 2 and 3 of Diewert (1992). For subsequent surveys of the axiomatic approach see Balk (1995) (2008).
[9] Multilateral index number theory refers to the case where there are more than two situations whose prices and quantities need to be aggregated.
[10] The problems caused by missing prices and quantities will be addressed in Chapters 7 and 8.
[11] Notation: $p >> 0_N$ means each component of p is positive, $p \geq 0_N$ means each component of p is nonnegative, $p > 0_N$ means $p \geq 0_N$ and $p \neq 0_N$ and $p\cdot q \equiv \Sigma_{n=1}^{N} p_nq_n$ where $p \equiv [p_1,\ldots,p_N]$ and $q \equiv [q_1,\ldots,q_N]$.

(10) $p^0 > 0_N$; $p^1 > 0_N$; $q^0 > 0_N$; $q^1 > 0_N$; $p^0 \cdot q^0 > 0$; $p^1 \cdot q^1 > 0$; $p^0 \cdot q^1 > 0$; $p^1 \cdot q^0 > 0$.

In the remainder of this chapter, we assume that p^0,p^1,q^0,q^1 satisfy either the strict positivity conditions, $p^0 >> 0_N$; $p^1 >> 0_N$; $q^0 >> 0_N$; $q^1 >> 0_N$, or the weaker conditions (10) above.[12]

In this section, the value ratio decomposition approach to index number theory will be taken; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indexes equals the value ratio between the two periods.[13] Thus, throughout this chapter, it is assumed that P and Q satisfy the following *product test*:

(11) $V^1/V^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$

The period t values, V^t, for t = 0,1 are defined by (1) above. Equation (11) means that as soon as the functional form for the price index P is determined, then (11) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that if a reasonable test is imposed on the quantity index Q, then (11) can be used to translate this test on the quantity index into a corresponding test on the price index P.[14]

If N = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0, the single price ratio, and a natural candidate for Q is q_1^1/q_1^0, the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, some twenty tests are listed that turn out to characterize the Fisher ideal price index. If it is desired to set $q^0 = q^1$, the common quantity vector is denoted by q; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

The first two tests are not very controversial and so they will not be discussed in detail.

T1: *Positivity* [15]: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity* [16]: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

The next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test* [17]: $P(p,p,q^0,q^1) = 1$.

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[12] Test T14 requires the additional assumption of strict positivity of the base period prices; i.e., T14 requires that $p^0 >> 0_N$.

[13] See section 2 of Chapter 2 for more on this approach, which was initially due to Fisher (1911; 403) (1922).

[14] This observation was first made by Fisher (1911; 400-406). Vogt (1980) and Diewert (1992) also pursued this idea.

[15] Eichhorn and Voeller (1976, 23) suggested this test.

[16] Fisher (1922; 207-215) informally suggested the essence of this test.

[17] Laspeyres (1871; 308), Walsh (1901; 308) and Eichhorn and Voeller (1976; 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871), which does not satisfy this test. This test is also a special case of Fisher's (1911; 409-410) price proportionality test. This Test could be called the *strong identity test*. The corresponding *weak identity test* is $P(p,p,q,q) = 1$; i.e., if *both* prices and quantities are equal for the two periods under consideration, then the price index should equal unity. This version of the identity test is not controversial.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.

T4: *Fixed Basket or Constant Quantities Test* [18]: $P(p^0,p^1,q,q) = p^1\cdot q/p^0\cdot q$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i \equiv p^1\cdot q$ divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i \equiv p^0\cdot q$.

If the price index P satisfies Test T4 and P and Q jointly satisfy the product test, (11) above, then it is easy to show[19] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is also somewhat controversial since p^0 and p^1 are allowed to be different.[20]

The following four tests restrict the behavior of the price index P as the *scale* of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in Current Prices* [21]: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related *proportionality test* $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact, Walsh (1901, 385) noted that this last test implies the identity test, T3.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

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[18] The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67-68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921b; 543-544), and Bowley (1901, 227). Vogt and Barta (1997; 49) correctly observed that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indexes which Fisher (1911; 405) translated into a test for the price index using the product test (11).

[19] See Vogt (1980; 70).

[20] A weaker version of Tests 3 and 4 is the following test: T3*: $P(p,p,q,q) = 1$ for all $p > 0_N$ and $q > 0_N$. Obviously, if prices and quantities are identical in the two periods under consideration, it is very reasonable that the bilateral price index (and the companion quantity index) equal unity. However, if prices are identical across periods but the two quantity vectors are different, then it is not clear that a bilateral index number formula that uses quantity or value weights should equal 1. An example of a weighted bilateral index number formula that satisfies T3* but not T3 is *the unit value price index* defined by $P_{UV}(p^0,p^1,q^0,q^1) \equiv [p^1\cdot q^1/1_N\cdot q^1]/ [p^0\cdot q^0/1_N\cdot q^0]$. The properties of unit value price indexes will be studied in chapter 7.

[21] This test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24) and Vogt (1980, 68).

T6: *Inverse Proportionality in Base Period Prices* [22]: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ if $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*:

$$P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test[23] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (11), it can be seen that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities* [24]:

$$P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (11), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

The next five tests are *invariance* or *symmetry tests*. Fisher (1922; 62-63, 458-460) and Walsh (1901; 105) (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as will be done in section 4 below. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is due to Irving Fisher (1922; 63)[25] and it

[22] Eichhorn and Voeller (1976; 28) suggested this test.

[23] Fisher (1911; 405) proposed the related test $P_L(p^0,p^1,q^0,\lambda q^0) = P_L(p^0,p^1,q^0,q^0) \equiv p^1\cdot q^0/p^0\cdot q^0$.

[24] This test was proposed by Diewert (1992; 216).

is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

The next test asks that the index be invariant to changes in the units of measurement.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ \alpha_1^{-1} q_1^0,\ldots,\alpha_N^{-1} q_N^0;\ \alpha_1^{-1} q_1^1,\ldots,\alpha_N^{-1} q_N^1) =$$
$$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ q_1^0,\ldots,q_N^0;\ q_1^1,\ldots,q_N^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1863; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

The next test asks that the formula be invariant to the period chosen as the base period.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indexes fail this test; e.g., the Laspeyres (1871) price index, P_L defined by $P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0$ and the Paasche (1874) price index, P_P defined by $P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1$, both fail this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

The next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory.[26] However, these tests are quite consistent with the weighted stochastic approach to index number theory to be discussed later in this chapter.

T12: *Quantity Reversal Test* (quantity weights symmetry test): $P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

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[25] "This [test] is so simple as never to have been formulated. It is merely taken for granted and observed instinctively. Any rule for averaging the commodities must be so general as to apply interchangeably to all of the terms averaged." Irving Fisher (1922; 63).

[26] The economic approach to index number theory assumes that given prices (and income), households choose quantity vectors that maximize their welfare or utility. Thus when prices change, in general household consumption vectors will change. Thus Tests 12 and 13 are not consistent with the economic approach to bilateral index number theory.

The next test is the analogue to T12 applied to quantity indexes:

T13: *Price Reversal Test* (price weights symmetry test)[27]:
$[p^1 \cdot q^1/p^0 \cdot q^0]/P(p^0,p^1,q^0,q^1) = [p^0 \cdot q^1/p^1 \cdot q^0]/P(p^1,p^0,q^0,q^1)$.

Thus if we use (11) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

(12) $Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1)$.

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

T14: *Mean Value Test for Prices* [28]:

(13) $\min_n \{p_n^1/p_n^0 ; n = 1,...,N\} \leq P(p^0,p^1,q^0,q^1) \leq \max_n \{p_n^1/p_n^0 ; n = 1,...,N\}$.

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be interpreted as some sort of an average of the N price ratios, p_n^1/p_n^0, it seems essential that the price index P satisfy this test.

The next test is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities* [29]:

(14) $\min_n \{q_n^1/q_n^0 ; n = 1,...,N\} \leq [V^1/V^0]/P(p^0,p^1,q^0,q^1) \leq \max_n \{q_n^1/q_n^0 ; n = 1,...,N\}$.

where $V^t \equiv p^t \cdot q^t$ is the period t value for the aggregate defined by (1) above. Using the product test (11) to define the quantity index Q in terms of the price index P, it can be seen that T15 is equivalent to the following property for the associated quantity index Q:

(15) $\min_n \{q_n^1/q_n^0 ; n = 1,...,N\} \leq Q(p^0,p^1,q^0,q^1) \leq \max_n \{q_n^1/q_n^0 ; n = 1,...,N\}$.

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_n^1/q_n^0 of the individual quantities.

In section 4 of Chapter 2, it was argued that it was very reasonable to take an average of the Laspeyres and Paasche price indexes as a single "best" measure of overall price change. This point of view can be turned into a test:

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[27] This test was proposed by Diewert (1992; 218).

[28] In the present context, this test seems to have been first proposed by Eichhorn and Voeller (1976; 10). Samuelson (1947) and Pollak (1971) showed that this test was satisfied by the Konüs true cost of living index which will be considered in Chapter 5. Note that this test requires that $p^0 >> 0_N$ so that all of the price ratios p_n^1/p_n^0 are well defined.

[29] This test was proposed by Diewert (1992; 219). Note that this test requires that $q^0 >> 0_N$ so that the quantity ratios q_n^1/q_n^0 are well defined.

T16: *Paasche and Laspeyres Bounding Test* [30]: The price index P lies between the Laspeyres and Paasche indexes, $P_L \equiv p^1 \cdot q^0/p^0 \cdot q^0$ and $P_P \equiv p^1 \cdot q^1/p^0 \cdot q^1$.

A test could be proposed where the implicit quantity index Q that corresponds to P via (11) is to lie between the Laspeyres and Paasche quantity indexes, Q_P and Q_L, defined as follows:

(16) $Q_L(p^0,p^1,q^0,q^1) \equiv p^0 \cdot q^1/p^0 \cdot q^0$; $Q_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/p^1 \cdot q^0$.

However, the resulting test turns out to be equivalent to test T16.

The final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^{1*},q^0,q^1)$ if $p^1 < p^{1*}$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^{0*},p^1,q^0,q^1)$ if $p^0 < p^{0*}$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0. This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^{1*}$, then

(17) $[p^1 \cdot q^1/p^0 \cdot q^0]/P(p^0,p^1,q^0,q^1) < [p^1 \cdot q^{1*}/p^0 \cdot q^0]/P(p^0,p^1,q^0,q^{1*})$.

T20: *Monotonicity in Base Quantities*: if $q^0 < q^{0*}$, then

(18) $[p^1 \cdot q^1/p^0 \cdot q^0]/P(p^0,p^1,q^0,q^1) > [p^1 \cdot q^1/p^0 \cdot q^{0*}]/P(p^0,p^1,q^{0*},q^1)$.

Let Q be the implicit quantity index that corresponds to P using (11). Then it is found that T19 translates into the following inequality involving Q:

(19) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^{1*})$ if $q^1 < q^{1*}$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(20) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^{0*},q^1)$ if $q^0 < q^{0*}$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

[30] Bowley (1901; 227) and Fisher (1922; 403) both endorsed this property for a price index.

This concludes the listing of tests. In the next section, we ask whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all twenty tests.

4. The Fisher Ideal index and the Test Approach

It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1-T20 is the Fisher ideal price index P_F defined as the geometric mean of the Laspeyres and Paasche indexes:[31]

$$(21)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)P_P(p^0,p^1,q^0,q^1)]^{1/2}$$

where $P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0$ and $P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1$.

To prove this assertion, it is relatively straightforward to show that the Fisher index satisfies all 20 tests. The more difficult part of the proof is to show that it is the *only* index number formula which satisfies these tests. This part of the proof follows from the fact that if P satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\begin{aligned}(22)\ [p^1\cdot q^1/p^0\cdot q^0]/[p^0\cdot q^1/p^1\cdot q^0] &= P(p^0,p^1,q^0,q^1)/P(p^1,p^0,q^0,q^1) \\ &= P(p^0,p^1,q^0,q^1)/P(p^1,p^0,q^1,q^0) && \text{using T12, the quantity reversal test} \\ &= P(p^0,p^1,q^0,q^1)P(p^0,p^1,q^0,q^1) && \text{using T11, the time reversal test.}\end{aligned}$$

Now take positive square roots on both sides of (22) and it can be seen that the left hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by (21) and the right hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

The quantity index that corresponds to the Fisher price index using the product test (11) is Q_F, the Fisher quantity index, defined as follows:

$$(23)\ Q_F(p^0,p^1,q^0,q^1) \equiv [V^1/V^0]/P_F(p^0,p^1,q^0,q^1) = [Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)]^{1/2}$$

where the Laspeyres and Paasche quantity indexes, Q_L and Q_P, are defined by (16). Thus the Fisher quantity index that corresponds via the product test (11) to the Fisher price index is also equal to the geometric mean of the Laspeyres and Paasche quantity indexes.

It turns out that P_F satisfies yet another test, T21, which was Irving Fisher's (1921; 534) (1922; 72-81) *third reversal test* (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):

$$(24)\ P(p^0,p^1,q^0,q^1)P(q^0,q^1,p^0,p^1) = V^1/V^0.$$

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if the roles of prices and quantities are reversed, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1/V^0. The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test. However, if

[31] See Diewert (1992; 221).

one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ that satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by (21). Thus the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[32]

5. The Test Performance of Other Indexes

The Fisher price index P_F satisfies all 20 of the tests listed in section 3 above. Which tests do other commonly used price indexes satisfy? The Laspeyres, Paasche and Fisher price indexes, P_L, P_P and P_F, have been defined above. Two other indexes which played a prominent role in Chapter 2 were the Walsh and Törnqvist indexes defined as follows:

(25) $P_W(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^1 / \Sigma_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^0$;
(26) $P_T(p^0,p^1,q^0,q^1) \equiv \Pi_{n=1}^N [p_n^1/p_n^0]^{(1/2)s_n^0+(1/2)s_n^1}$;

where $s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t$ for $n = 1,\ldots,N$ and $t = 0,1$. Note that in order that $P_T(p^0,p^1,q^0,q^1)$ be well defined, it is required that $p^0 \gg 0_N$; i.e., each base period price p_n^0 must be positive.

Straightforward computations show that the Paasche and Laspeyres price indexes, P_L and P_P, fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seem at first sight to be quite good. However, the failure of the time reversal test, T11, is a severe limitation associated with the use of these indexes.

The Walsh price index, P_W, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

Finally, the Törnqvist price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the 4 monotonicity tests T17 to T20. Thus the Törnqvist index is subject to a rather high failure rate from the viewpoint of this axiomatic approach to index number theory.[33]

The tentative conclusion that can be drawn from the above results is that from the viewpoint of this particular bilateral test approach to index numbers, the Fisher ideal price index P_F appears to be "best" since it satisfies all 20 tests.[34] The Paasche and Laspeyres indexes are next best if we treat each test as being equally important. However, both of these indexes fail the very important time reversal test. The remaining two indexes, the Walsh and Törnqvist price indexes, both

[32] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985) (1995).
[33] However, it will be shown later in Chapter 5 that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that are subject to relatively smooth trends. Hence under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.
[34] This assertion needs to be qualified: there are many other tests that we have not discussed and price statisticians could differ on the importance of satisfying various sets of tests. Some references which discuss other tests are Auer (2002), Eichhorn and Voeller (1976), Balk (1995) (2008) and Vogt and Barta (1997). In section 7 below, it is shown that the Törnqvist index is "best" for a different set of axioms.

satisfy the time reversal test but the Walsh index emerges as being "better" since it passes 16 of the 20 tests whereas the Törnqvist only satisfies 11 tests. However, in section 7 below, we will change the axiomatic framework and in this new framework, the Törnqvist price index will emerge as "best" in this alternative framework. Before this new framework is considered, one important additional test in the present axiomatic framework will be discussed in the following section.

6. The Circularity Test

If the identity test T3 is true, then the time reversal test T11 can be rewritten as follows:

(27) $1 = P(p^0,p^0,q^0,q^0) = P(p^0,p^1,q^0,q^1)P(p^1,p^0,q^1,q^0)$.

Thus if one starts out with the prices p^0 in period 0 and go to the prices p^1 in period 1 but then return to the prices of period 0 in period 2, and if the tests T3 and T11 are satisfied, then the product of the price movement from period 0 to 1, $P(p^0,p^1,q^0,q^1)$, and the price movement from period 1 to 2, $P(p^1,p^0,q^1,q^0)$, turns out to equal 1, indicating that the *chained price index* in period 2 has returned to its period 0 level of 1. An obvious generalization of (27) would be to replace the assumption that the period 2 price and quantity vectors in the above formula are the same as the period 0 price and quantity vectors, p^0 and q^0, and allow for arbitrary period 2 price and quantity vectors, p^2 and q^2. With this replacement, (27) becomes:

(28) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

If an index number formula P satisfies (28), then we say that P satisfies the *circularity test*.[35]

What is the meaning of (28)? The index number on the left hand side of (28), compares prices in period 2 directly with prices in period 0 and $P(p^0,p^2,q^0,q^2)$ is called the *fixed base price index* for period 2. The *chained price index* for period 2, $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$, on the right hand side of (28) compares prices in period 2 with those in period 0 by first comparing prices in period 1 with those in period 0 (this is the *chain link index* $P(p^0,p^1,q^0,q^1)$) and multiplies that index by the chain link index that compares prices in period 2 to those of period 1, $P(p^1,p^2,q^1,q^2)$. If the index number formula P satisfies the circularity test (28), then it does not matter whether we use the *chained index* (the right hand side of (28)) to compare prices in period 2 with those of the base period 0 or if we use the *fixed base index* (the left hand side of (28)): *we get the same answer either way*. Obviously, it would be desirable if one could find an index number formula that satisfied the circularity test and had satisfactory axiomatic properties with respect to the other tests that have been considered.

Unfortunately, it turns out that index number formulae that satisfy the circularity test have other properties that make it unsatisfactory. Consider the following result:

Proposition 1: Suppose that the index number formula P satisfies the following tests: T1 (positivity), T2 (continuity), T3 (identity), T5 (proportionality in current prices), T10 (commensurability) and T17 (monotonicity in current prices) in addition to the circularity test

[35] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

above. Then P must have the following functional form due originally to Konüs and Byushgens[36] (1926; 163-166):[37]

(29) $P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{\alpha_n}$

where the N constants α_n satisfy the restrictions $\sum_{n=1}^{N} \alpha_n = 1$ and $\alpha_n > 0$ for n = 1,…,N.

A proof of the above result is in the Appendix. The above result says that under fairly weak regularity conditions, *the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios*, the weights being constant through time. The α_n weights could be chosen to be the average expenditure shares on the N commodities over the time period when the index number formula is being used. If expenditure shares are close to being constant over the sample period, the resulting weighted geometric mean index defined by (29) will be a perfectly good index. However, if there are strong (divergent) trends in expenditure shares and strong (divergent) trends in the prices of the N commodities, then the index will not have representative weights over the entire sample period and thus will not be able to adequately represent price movements over the entire sample period.[38]

An interesting special case of the family of indexes defined by (29) occurs when the weights α_n are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

(30) $P_J(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{1/N}$.

The problem with the indexes defined by Konüs and Byushgens and Jevons is that the individual price ratios, p_n^1/p_n^0, have weights (either α_n or 1/N) that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights for commodity n are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indexes are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.[39]

[36] Konüs and Byushgens (1926) showed that the index defined by (29) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1971). The concept of an exact index number formula will be explained in Chapter 5 when the economic approach to index number theory is studied.
[37] See also Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). Proofs of related results can be found in Funke, Hacker and Voeller (1979) and Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…" Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."
[38] This lack of representative weights problem will be particularly acute if there are disappearing and newly appearing commodities. However, note that Proposition 1 does not deal adequately with this problem since it is assumed that all prices and quantities are positive for the two periods under consideration. The problems associated with missing prices and quantities will be addressed in Chapters 5, 7 and 8.
[39] However, as mentioned above, if the expenditure shares are not changing much from period to period (or better yet, are constant), then by choosing the α_n to be these constant expenditure shares, the Konüs and Byushgens price index will reduce to the Törnqvist price index P_T, defined by (26) above, which has good statistical properties.

Proposition 1 above is a result that applies to bilateral index number functions of the form $P(p^0,p^1,q^0,q^1)$, where it is assumed that all prices and quantities can vary independently. However, the Lowe index defined in Chapter 2 does not fit into this framework. Recall that the *Lowe index* is defined as follows:

(31) $P_{Lo}(p^0,p^1,q) \equiv \Sigma_{n=1}^N p_n^1 q_n / \Sigma_{n=1}^N p_n^0 q_n = p^1 \cdot q / p^0 \cdot q$

where $p^0 \equiv [p_1^0,...,p_N^0]$ and $p^1 \equiv [p_1^1,...,p_N^1]$ are the price vectors for periods 0 and 1 and $q \equiv [q_1,...,q_N]$ is a representative quantity vector. It can be seen that the Lowe index does not fit into the axiomatic framework that was developed for index number formulae of the form $P(p^0,p^1,q^0,q^1)$.

It is possible to adapt many of the tests listed in section 3 above to a new index number framework that looks at the axiomatic properties of indexes of the form $P(p^0,p^1,q)$. Thus the section 3 tests T1-T20 that can be adapted to this new framework have the following counterpart tests:

T1: *Positivity*: $P(p^0,p^1,q) > 0$.[40]

T2: *Continuity*: $P(p^0,p^1,q)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,q) = 1$.

T5: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,q) = \lambda P(p^0,p^1,q)$ for $\lambda > 0$.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,q) = \lambda^{-1} P(p^0,p^1,q)$ if $\lambda > 0$.

T7: *Invariance to Proportional Changes in Quantities*:

$$P(p^0,p^1,\lambda q) = P(p^0,p^1,q) \text{ for all } \lambda > 0.$$

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^*) = P(p^0,p^1,q)$$

where p^{t*} denotes a permutation of the components of the vector p^t for $t = 0,1$ and q^* denotes the same permutation of the components of q.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,...,\alpha_N p_N^0; \alpha_1 p_1^1,...,\alpha_N p_N^1; \alpha_1^{-1} q_1,...,\alpha_N^{-1} q_N) = P(p^0,p^1,q).$$

T11: *Time Reversal Test*: $P(p^0,p^1,q) = 1/P(p^1,p^0,q)$.

T14: *Mean Value Test for Prices*:

$$\min_n \{p_n^1/p_n^0 \,;\, n = 1,...,N\} \le P(p^0,p^1,q) \le \max_n \{p_n^1/p_n^0 \,;\, n = 1,...,N\}.$$

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q) < P(p^0,p^{1*},q)$ if $0_N < p^1 < p^{1*}$ and $q >> 0_N$.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q) > P(p^{0*},p^1,q)$ if $0_N < p^0 < p^{0*}$ and $q >> 0_N$.

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[40] Unless otherwise specified, the domain of definition for $P(p^0,p^1,q)$ is $p^0 > 0_N$, $p^1 > 0_N$, $q > 0_N$, $p^0 \cdot q > 0$ and $p^1 \cdot q > 0$.

Thus 12 of the 20 tests listed in section 3 have counterparts that can be applied to indexes of the form $P(p^0,p^1,q)$. The counterpart to the circularity test in the present framework is the following test:

T22: *Circularity*: $P(p^0,p^2,q) = P(p^0,p^1,q)P(p^1,p^2,q)$.

A question of interest is: are the above tests sufficient to determine the functional form for $P(p^1,p^2,q)$? Using the positivity test T1, rewrite the circularity test T22 in the following form:

(32) $P(p^1,p^2,q) = P(p^0,p^2,q)/P(p^0,p^1,q)$.

Now hold p^0 constant at some fixed value, say $p^* >> 0_N$ and define the function $f(p,q)$ as follows:

(33) $f(p,q) \equiv P(p^*,p,q) > 0$ for all $p>>0_N$ and $q >> 0_N$

where the positivity of $f(p,q)$ follows from T1. Substituting definition (33) back into (32) gives us the following representation for $P(p^1,p^2,q)$:

(34) $P(p^1,p^2,q) = f(p^2,q)/f(p^1,q)$.

Thus in this axiomatic framework, the price index $P(p^1,p^2,q)$ is equal to the price level for period 2, $f(p^2,q)$, divided by the price level for period 1,$f(p^1,q)$. The function $f(p,q)$ determines the functional form for the price index. Various properties on f can be imposed so that the above tests are satisfied by the price index function, $P(p^1,p^2,q)$. Imposing continuity on $f(p,q)$ will ensure that test T2 is satisfied. The identity test T3 will automatically be satisfied by a $P(p^1,p^2,q)$ defined as $f(p^2,q)/f(p^1,q)$. Imposing the linear homogeneity property $f(\lambda p,q) = \lambda f(p,q)$ for all $\lambda > 0$ will ensure that $P(p^1,p^2,q)$ will satisfy tests T5 and T6. Imposing the linear homogeneity property $f(p,\lambda q) = \lambda f(p,q)$ for all $\lambda > 0$ will ensure that $P(p^1,p^2,q)$ will satisfy test T7. Imposing the commodity reversal and commensurability tests on $f(p,q)$ will ensure that $P(p^1,p^2,q)$ will satisfy tests T9 and T10. The time reversal test T11 will automatically be satisfied by a $P(p^1,p^2,q)$ defined as $f(p^2,q)/f(p^1,q)$. Simple conditions on $f(p,q)$ that will ensure that $P(p^1,p^2,q)$ will satisfy the mean value test T14 are difficult to determine. If $f(p,q)$ is monotonically increasing in the components of p, then $P(p^1,p^2,q)$ will satisfy the monotonicity tests T17 and T18. In general, it appears that the tests listed above are not sufficient to determine the functional form for $f(p,q)$ and hence the listed tests (including the circularity test) do not determine a unique functional form for $P(p^1,p^2,q)$.

Recall the test T4 from section 3, the *Fixed Basket or Constant Quantities Test*, which was the following test:

(35) $P(p^0,p^1,q,q) = p^1\cdot q/p^0\cdot q$.

This test is relevant in the present context where we have only a single reference quantity vector q. Test T4 from section 3 suggests that if the quantities purchased in periods 0 and 1 were identical, then the price index should equal $p^1\cdot q/p^0\cdot q$ where q is the common quantity vector; i.e., $q = q^0 = q^1$. Thus if $q^0 = q^1$, then the representative quantity vector q is obviously this common quantity vector so in this case, $P(p^0,p^1,q)$ should equal $p^1\cdot q/p^0\cdot q$.[41] This suggests that even if q^0 is not equal to q^1, the functional form for $P(p^0,p^1,q)$ should be set equal to $p^1\cdot q/p^0\cdot q$, since this

[41] Thus $f(p,q) = p\cdot q$.

functional form will register the correct result for a price index when the quantity vectors for periods 0 and 1 are identical (or proportional). Thus the application of test T4 to the present context pins down the functional form for $P(p^0,p^1,q)$:

(36) $P(p^0,p^1,q) \equiv p^1 \cdot q/p^0 \cdot q \equiv P_{Lo}(p^0,p^1,q)$.

Thus the new axiomatic framework leads to the Lowe index, $P_{Lo}(p^0,p^1,q)$, as being "best" in this framework. It is straightforward to show that the Lowe index satisfies Tests T1, T2, T3, T5, T6, T7, T9, T10, T11, T14, T17, T18 and T22; i.e., it satisfies all of the modified tests that were listed in this section.

The Lowe index works well if the quantities demanded grow in a proportional manner (or approximately proportional manner) over time. But if prices and quantities have divergent trends over time, the Lowe index will be subject to *substitution bias*; i.e., it will tend to register higher rates of inflation than the economic indexes to be considered in Chapter 5 which deal more adequately with substitution bias.

As was seen above, it is possible to find index number formulae (see (29) and (36)) that satisfy the circularity test but the resulting indexes are not entirely satisfactory.

In the following section, another axiomatic framework for bilateral index number formulae will be discussed.

7. An Alternative Axiomatic Approach to Bilateral Index Number Theory

One of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, $r_n \equiv p_n^1/p_n^0$.[42] This is equivalent to using an axiomatic approach to try to determine the "best" index of the form $P(r,v^0,v^1)$, where v^0 and v^1 are the vectors of expenditures on the N commodities during periods 0 and 1.[43] However, initially, rather than starting with indexes of the form $P(r,v^0,v^1)$, indexes of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework is more comparable to the first bilateral axiomatic framework taken in sections 3 and 4 above. As will be seen below, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P^*(r,v^0,v^1)$.

[42] Fisher also took this point of view when describing his approach to index number theory: "An index number of the prices of a number of commodities is an average of their price relatives. This definition has, for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances." Irving Fisher (1922; 3). However, in setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indexes written as functions of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1; i.e., he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indexes of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indexes and, as was seen in Chapter 2, each of these indexes can be written as expenditure share weighted averages of the N price relatives, $r_n \equiv p_n^1/p_n^0$.

[43] Chapter 3 in Vartia (1976) considered a variant of this axiomatic approach.

Recall that the product test (11) was used in order to define the quantity index, $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0P(p^0,p^1,q^0,q^1)$, that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; i.e., given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, then the corresponding *implicit quantity index* can be defined in terms of P as follows:

(37) $Q(p^0,p^1,v^0,v^1) \equiv [\sum_{n=1}^N v_n^1]/[(\sum_{n=1}^N v_n^0)\,P(p^0,p^1,v^0,v^1)]$.

In sections 3 and 4 above, the price and quantity indexes $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ were determined *jointly*; i.e., not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$ but they were also imposed on $Q(p^0,p^1,q^0,q^1)$ and the product test (11) was used to translate these tests on Q into tests on P. In what follows, only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this form. Thus there is a parallel theory for quantity indexes of the form $Q(q^0,q^1,v^0,v^1)$ where it is attempted to find the "best" value weighted average of the quantity relatives, q_n^1/q_n^0.[44]

For the most part, the tests which will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this section are counterparts to the tests that were imposed on the price index $P(p^0,p^1,q^0,q^1)$ in section 3 above. It will be assumed that every component of each price and value vector is positive; i.e., $p^t >> 0_N$ and $v^t >> 0_N$ for $t = 0,1$. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

The first two tests are straightforward counterparts to the corresponding tests in section 3.

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,v^0,v^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,v^0,v^1 changes.

T4: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. This test is the counterpart to test T5 in section 3 above.

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[44] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1/[\sum_{n=1}^N v_n^0\, Q(q^0,q^1,v^0,v^1)]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus the axiomatic approach to be developed in this section generates separate "best" price and quantity indexes whose product does not equal the value ratio in general. This is a disadvantage of this third axiomatic approach to bilateral indexes compared to the first approach studied in sections 3 and 4 above.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0. This test is the counterpart to test T6 in section 3.

The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to Proportional Changes in Current Period Values*:
$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$.

That is, if current period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 1 value vector v^1.

T7: *Invariance to Proportional Changes in Base Period Values*:
$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$.

That is, if base period values are all multiplied by the number λ, then the price index remains unchanged. Hence the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 0 value vector v^0.

T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors v^0 and v^1. Using test T6 with $\lambda = 1/\sum_{i=1}^N v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^N v_i^0$, it can be seen that P has the following property:

(38) $P(p^0,p^1,v^0,v^1) = P(p^0,p^1,s^0,s^1)$

where s^0 and s^1 are the vectors of expenditure shares for periods 0 and 1; i.e., the ith component of s^t is $s_i^t \equiv v_i^t/\sum_{k=1}^N v_k^t$ for $t = 0,1$ and $i = 1,...,N$. Thus the tests T6 and T7 imply that the price index function P is a function of the two price vectors p^0 and p^1 and the two vectors of expenditure shares, s^0 and s^1.

Walsh suggested the spirit of tests T6 and T7 as the following quotation indicates:

"What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [i.e., the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered." Correa Moylan Walsh (1901; 104).

Walsh also realized that weighting the ith price relative r_i by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

"At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*." Correa Moylan Walsh (1901; 104-105).

As a solution to the above weighting problem, Walsh (1901; 202) (1921a; 97) proposed the following *geometric Walsh price index*:

(39) $P_{GW}(p^0,p^1,v^0,v^1) \equiv \prod_{n=1}^N [p_n^1/p_n^0]^{w_n}$

where the nth weight in the above formula was defined as

(40) $w_n \equiv (v_n^0 v_n^1)^{1/2}/\sum_{i=1}^N (v_i^0 v_i^1)^{1/2} = (s_n^0 s_n^1)^{1/2}/\sum_{i=1}^N (s_i^0 s_i^1)^{1/2}$; $n = 1,\ldots,N.$

The second equation in (40) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, s^0 and s^1; i.e., $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree 0 in the components of the value vectors v^0 and v^1 and so $P_{GW}(p^0,p^1,v^0,v^1) = P_{GW}(p^0,p^1,s^0,s^1)$. Thus Walsh came very close to deriving the Törnqvist index defined earlier by (26).[45]

The next 5 tests are *invariance* or *symmetry tests* and 4 of them are direct counterparts to similar tests in section 3 above. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

T8: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):
$P(p^{0*},p^{1*},v^{0*},v^{1*}) = P(p^0,p^1,v^0,v^1)$

where p^{t*} denotes a permutation of the components of the vector p^t and v^{t*} denotes the same permutation of the components of v^t for $t = 0,1$.

The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to Changes in the Units of Measurement* (commensurability test):
$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1)$
$= P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1)$ for all $\alpha_1 > 0, \ldots, \alpha_N > 0$.

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, v_i^t, does not change if the unit by which commodity i is measured changes.

The last test has a very important implication. Let $\alpha_1 = 1/p_1^0, \ldots, \alpha_N = 1/p_N^0$ and substitute these values for the α_i into the definition of the test. The following equation is obtained:

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[45] It is evident that Walsh's geometric price index will closely approximate the Törnqvist index using normal time series data. More formally, regarding both indexes as functions of p^0,p^1,v^0,v^1, it can be shown that $P_{GW}(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (i.e., $p^0 = p^1$) and expenditure (i.e., $v^0 = v^1$) point.

(41) $P(p^0,p^1,v^0,v^1) = P(1_N,r,v^0,v^1) \equiv P^*(r,v^0,v^1)$

where 1_N is a vector of ones of dimension N and r is a vector of the price relatives; i.e., the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of 4N variables, can be written as a function of 3N variables, $P^*(r,v^0,v^1)$, where r is the vector of price relatives and $P^*(r,v^0,v^1)$ is defined as $P(1_N,r,v^0,v^1)$.

The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time Reversal Test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section).

The next test is a variant of the *circularity test*, that was introduced in section 6 above.

T11: *Transitivity in Prices for Fixed Value Weights*: $P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s) = P(p^0,p^2,v^r,v^s)$.

In this test, the expenditure weighting vectors, v^r and v^s, are held constant while making all price comparisons. However, given that these weights are held constant, then the test asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. Obviously, this test is a many commodity counterpart to a property that holds for a single price relative.

The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Quantity Weights Symmetry Test*: $P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$.

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that if values are used to weight the prices in the index number formula, then the period 0 values v^0 and the period 1 values v^1 must enter the formula in a symmetric or even handed manner.

The next test is a *mean value test*.

T13: *Mean Value Test*: $\min_i (p_i^1/p_i^0 : i =1,...,N) \leq P(p^0,p^1,v^0,v^1) \leq \max_i (p_i^1/p_i^0 : i = 1,...,N)$.

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the N price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

The next two tests in this section are *monotonicity tests*; i.e., how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors p^0 and p^1 increases.

T14: *Monotonicity in Current Prices*: $P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase (holding the value vectors fixed), so that $P(p^0,p^1,v^0,v^1)$ is increasing in the components of p^1 for fixed p^0, v^0 and v^1.

T15: *Monotonicity in Base Prices*: $P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,v^0,v^1)$ is decreasing in the components of p^0 for fixed p^1, v^0 and v^1.

The above tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (39) and the Törnqvist index $P_T(p^0,p^1,v^0,v^1)$ defined by (26)[46] satisfy all of the above axioms. Thus at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

The tests proposed thus far do not specify exactly how the expenditure share vectors s^0 and s^1 are to be used in order to weight, for example, the first price relative, p_1^1/p_1^0. The next test says that only the expenditure shares s_1^0 and s_1^1 pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, p_1^1 and p_1^0.

T16: *Own Share Price Weighting*:

(42) $P(p_1^0,1,\ldots,1;p_1^1,1,\ldots,1;v^0;v^1) = f(p_1^0,p_1^1,v_1^0/\sum_{n=1}^N v_n^0,\ v_1^1/\sum_{n=1}^N v_n^1)$.

Note that $v_1^t/\sum_{k=1}^N v_k^t$ equals s_1^t, the expenditure share for commodity 1 in period t. The above test says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of 2 + 2N variables is actually only a function of 4 variables.[47]

If test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

(43) $P(1,\ldots,1,p_i^0,1,\ldots,1;1,\ldots,1,p_i^1,1,\ldots,1;v^0;v^1) = f(p_i^0,p_i^1,v_i^0/\sum_{n=1}^N v_n^0,\ v_i^1/\sum_{n=1}^N v_n^1)$; $i = 1,\ldots,N$.

Equation (43) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of Price Change with Tiny Value Weights*:

(44) $P(p_1^0,1,\ldots,1;\ p_1^1,1,\ldots,1;\ 0,v_2^0,\ldots,v_N^0;\ 0,v_2^1,\ldots,v_N^1) = 1$.

The test T17 says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are zero in the two periods but the

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[46] The share weights s_n^t in definition (26) can be rewritten as $v_n^t/v^t\cdot 1_N$ for n = 1,...,N and t = 0,1. Thus $P_T(p^0,p^1,q^0,q^1)$ can be rewritten in the form $P_T(p^0,p^1,v^0,v^1)$.

[47] In the economics literature, axioms of this type are known as separability axioms.

expenditures on the other commodities are arbitrarily given, then the index is equal to 1.[48] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for i = 1,…,N:

(45) $P(1,\ldots,1,p_i^0,1,\ldots,1;\ 1,\ldots,1,p_i^1,1,\ldots,1;\ v_1^0,\ldots,v_{i-1}^0,0,v_{i+1}^0,\ldots,v_N^0;\ v_1^1,\ldots,v_{i-1}^1,0,v_{i+1}^1,\ldots,v_N^1) = 1.$

Equation (45) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, and the expenditures on commodity i are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

This completes the listing of tests for the weighted average of price relatives approach to bilateral index number theory. It turns out that the above tests are sufficient to imply a specific functional form for the price index as will be seen in the next section.

8. The Törnqvist Price Index and the Alternative Approach to Bilateral Indexes

It turns out that the Törnqvist price index is the only index that satisfies the axioms in the previous section.

Proposition 2: If the number of commodities N exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed in section 7 above, then P must be the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$ defined by (26).[49]

Thus the 17 properties or tests listed in section 7 provide an axiomatic characterization of the Törnqvist price index, just as the 20 tests listed in section 3 provided an axiomatic characterization for the Fisher ideal price index. For a proof of Proposition 2, see the Appendix.

Obviously, there is a parallel axiomatic theory for quantity indexes of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, q^0 and q^1, as well as on the corresponding two expenditure vectors, v^0 and v^1. Thus if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist quantity index $Q_T(q^0,q^1,v^0,v^1)$, whose logarithm is defined as follows:

(46) $\ln Q_T(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1)\ln(q_n^1/q_n^0)$

where as usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t/\sum_{k=1}^N v_k^t$ for i = 1,…,N and t = 0,1.

Unfortunately, the implicit Törnqvist Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist quantity index Q_T defined by (46) using the product test is *not* equal to the direct

[48] Strictly speaking, since all prices and values are required to be positive, the left hand side of (44) should be replaced by the limit as the commodity 1 values, v_1^0 and v_1^1, approach 0.

[49] The Törnqvist price index satisfies all 17 tests but the proof in the Appendix did not use all of these tests to establish the result in the opposite direction: tests 5, 13, 15 and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist price index. For alternative characterizations of the Törnqvist Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined earlier by (26). The product test equation that defines P_{IT} in the present context is given by the following definition:

(47) $P_{IT}(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1/[\sum_{n=1}^N v_n^0 Q_T(q^0,q^1,v^0,v^1)] = v^1\cdot 1_N/[v^0\cdot 1_N Q_T(q^0,q^1,v^0,v^1)]$.

The fact that the direct Törnqvist price index P_T is not in general equal to the implicit Törnqvist Theil price index P_{IT} defined by (47) is a bit of a disadvantage compared to the axiomatic approach outlined in sections 3 and 4 above, which led to the Fisher ideal price and quantity indexes as being "best". Using the Fisher approach meant that it was not necessary to decide whether one wanted a "best" price index or a "best" quantity index: the theory outlined in sections 3 and 4 determined both indexes simultaneously. However, in the Törnqvist approach outlined in this section, it is necessary to *choose* whether one wants a "best" price index or a "best" quantity index.[50]

Other tests are of course possible. A counterpart to Test T16 in section 3, the Paasche and Laspeyres bounding test, is the following *geometric Paasche and Laspeyres bounding test*:

(48) $P_{GL}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GP}(p^0,p^1,v^0,v^1)$ or
$P_{GP}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GL}(p^0,p^1,v^0,v^1)$

where the logarithms of the *geometric Laspeyres* and *geometric Paasche* price indexes, P_{GL} and P_{GP}, are defined as follows:

(49) $\ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^N s_n^0 \ln(p_n^1/p_n^0)$;
(50) $\ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^N s_n^1 \ln(p_n^1/p_n^0)$

It can be shown that the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$ defined by (26) satisfies the geometric Laspeyres and Paasche bounding test but the geometric Walsh price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (39) does not satisfy it.

The geometric Paasche and Laspeyres bounding test was not included as a primary test in section 7 because, a priori, it was not known what form of averaging of the price relatives (e.g., geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (48) is an appropriate one if it has been decided that geometric averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indexes correspond to "extreme" forms of value weighting in the context of geometric averaging and it is natural to require that the "best" price index lie between these extreme indexes.

Walsh (1901; 408) pointed out a problem with his geometric price index P_{GW} defined by (39), which also applies to the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$: these geometric type indexes do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be the *Lowe* (1823) *index*, which is the ratio of the costs of purchasing the constant basket during the two periods.[51] Put another way, the geometric indices P_{GW} and P_T do not satisfy the fixed basket test, T4 in section 3 above.

[50] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist price indexes in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section. For more on Hillinger's approach to index number theory, see Hillinger (2002).

[51] Of course, the Fisher ideal index does have this property and gives the "right" answer when q^1 is equal or proportional to q^0.

There is one additional test that should be mentioned. Fisher (1911; 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows:

"A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero." Irving Fisher (1911; 401).

In the present context, this test could be interpreted as the following one: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p,v^0,v^1)$ should not tend to zero or plus infinity. However, with this interpretation of the test, which regards the values v_i^t as remaining constant as the p_i^0 or p_i^1 tends to zero, none of the commonly used index number formulae would satisfy this test. Hence this test should be interpreted as a test that applies to price indexes $P(p^0,p^1,q^0,q^1)$ of the type that were studied in sections 3 and 4 above, which is how Fisher intended the test to apply. Thus Fisher's price determinateness test should be interpreted as follows: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p,q^0,q^1)$ should not tend to zero or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche and Fisher indexes satisfy this test but the Törnqvist price index will not satisfy this test. Thus when using the Törnqvist price index, *care must be taken to bound the prices away from zero in order to avoid a meaningless index number value*.

Walsh was aware that geometric average type indexes like the Törnqvist Theil price index P_T or Walsh's geometric price index P_{GW} defined by (39) become somewhat unstable[52] as individual price relatives become very large or small:

"Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i. e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably." Correa Moylan Walsh (1901; 373).

"In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together." Correa Moylan Walsh (1901; 404).

Weighing all of the arguments and tests presented in this chapter, there is a preference for the use of the Fisher ideal price index as a suitable target index for a statistical agency that wishes to use the axiomatic approach, but of course, opinions can differ on which set of axioms is the most appropriate to use in practice.

9. Defining Contributions to Overall Percentage Change for a Bilateral Index

Business analysts often want statistical agencies to provide decompositions of overall price change into explanatory components that reflect individual commodity price change. This decomposition problem can be defined more precisely as follows. A bilateral price index of the form $P(p^0,p^1,q^0,q^1)$ can be interpreted as the ratio of a period 1 price level, P^1, to a period 0 price level, P^0. Thus the *percentage change in the overall price level* is:

(51) $P(p^0,p^1,q^0,q^1) - 1 = [P^1/P^0] - 1 = [P^1 - P^0]/P^0$.

[52] That is, the index may approach zero or plus infinity.

The *percentage change in commodity price n* is $(p_n^1/p_n^0) - 1$ for $n = 1,...,N$.[53] The desired decomposition has the following form:

(52) $P(p^0,p^1,q^0,q^1) - 1 = \Sigma_{n=1}^N w_n[(p_n^1/p_n^0) - 1]$

where the w_n are *weighting factors* to be determined. The overall *contribution factor for commodity n* is defined as:

(53) $C_n \equiv w_n[(p_n^1/p_n^0) - 1]$; $n = 1,...,N$.

The problem is: how exactly are the weighting factors w_n to be determined? At the outset, it should be recognized that there need not be a unique determination for these weighting factors since the weighting factors are allowed to be functions of the 4N variables, p^0,p^1,q^0,q^1. The price index $P(p^0,p^1,q^0,q^1)$ will typically be a rather complicated function of the variables, p^0,p^1,q^0,q^1, and thus there can be many ways of decomposing $P(p^0,p^1,q^0,q^1) - 1$ into the form given by the right hand side of (52).[54]

The approach taken in this section is to use a simple first order Taylor series approximation to the index number formula to give us an approximate decomposition of the form (52). However, the suggested approximation requires an extra assumption; namely that the given index number formula, $P(p^0,p^1,q^0,q^1)$, can be rewritten in the form $P^*(r,s^0,s^1)$ where $r \equiv [r_1,...,r_N] = [p_1^1/p_1^0,...,p_N^1/p_N^0]$ and s^t is the usual vector of expenditure shares on the N commodities for period t for t = 0,1. Thus it is assumed that $P(p^0,p^1,q^0,q^1)$ can be expressed in the price ratio and expenditure share framework for bilateral indexes that was explained in section 7 above.[55] Definitions (54)-(58) show how the Laspeyres, Paasche, Fisher, Törnqvist and Walsh indexes can be expressed in the form $P^*(r,s^0,s^1)$:

(54) $P_L^*(r,s^0,s^1) \equiv \Sigma_{n=1}^N s_n^0 r_n$;
(55) $P_P^*(r,s^0,s^1) \equiv [\Sigma_{n=1}^N s_n^1 (r_n)^{-1}]^{-1}$;
(56) $P_F^*(r,s^0,s^1) \equiv [P_L^*(r,s^0,s^1)P_P^*(r,s^0,s^1)]^{1/2}$;
(57) $P_T^*(r,s^0,s^1) \equiv \Pi_{n=1}^N r_n^{(1/2)(s_n^0+s_n^1)}$;
(58) $P_W^*(r,s^0,s^1) \equiv \Sigma_{n=1}^N (s_n^0 s_n^1)^{1/2} (r_n)^{1/2} / \Sigma_{n=1}^N (s_n^0 s_n^1)^{1/2} (r_n)^{-1/2}$.

Using the new notation, the desired decomposition of overall percentage price change for $P(p^0,p^1,q^0,q^1)$ given by (52) can be rewritten as follows:

(59) $P^*(r,s^0,s^1) - 1 = \Sigma_{n=1}^N w_n(r_n - 1)$.

Assume that $P(p^0,p^1,q^0,q^1)$ satisfies the identity test, $P(p,p,q^0,q^1) = 1$ for all $p >> 0_N$, $q^0 >> 0_N$ and $q^1 >> 0_N$. Then the companion $P^*(r,s^0,s^1)$ will satisfy:

(60) $P^*(1_N,s^0,s^1) = 1$ for all $s^0 >> 0_N$ and $s^1 >> 0_N$

[53] In this section, it is assumed that all period 0 prices are positive so that the ratios p_n^1/p_n^0 are well defined.
[54] For example, see the alternative decompositions of the form (52) for the Fisher ideal index $P_F(p^0,p^1,q^0,q^1)$ that were obtained by Van IJzeren (1987), Ehemann, Katz and Moulton (2002), Diewert (2002) and Reinsdorf, Diewert and Ehemann (2002).
[55] See equations (41) above.

where 1_N is a vector of ones of dimension N.[56] Assuming that $P^*(r,s^0,s^1)$ is differentiable with respect to the components of r at $r = 1_N$, the *first order Taylor series approximation* to $P^*(r,s^0,s^1)$ around the point $r = 1_N$ is:

$$(61)\ P^*(r,s^0,s^1) \approx P^*(1_N,s^0,s^1) + \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1]$$
$$= 1 + \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1]$$

where the second line in (61) follows from (60). Rearranging (61) leads to the following approximate equality:

$$(62)\ P^*(r,s^0,s^1) - 1 \approx \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1].$$

Thus (62) has the same structure as (59) except (62) is only an approximate equality. The w_n weighting factors from (62) are the partial derivatives of $P^*(r,s^0,s^1)$ with respect to the components of r, evaluated at $r = 1_N$. The interpretation of these weighting factors is fairly straightforward.

Denote the nth weighting factor for the Laspeyres, Paasche, Fisher, Törnqvist and Walsh indexes as w_{Ln}, w_{Pn}, w_{Fn}, w_{Tn} and w_{Wn} respectively. Straightforward calculations of the respective partial derivatives show that these weighting factors are equal to the following expressions for n = 1,...,N:

$$(63)\ w_{Ln} = s_n^0 ;$$
$$(64)\ w_{Pn} = s_n^1 ;$$
$$(65)\ w_{Fn} = ½(s_n^0 + s_n^1) ;$$
$$(66)\ w_{Tn} = ½(s_n^0 + s_n^1) ;$$
$$(67)\ w_{Wn} = (s_n^0 s_n^1)^{1/2}/\Sigma_{i=1}^N (s_i^0 s_i^1)^{1/2}.$$[57]

The above weighting factors can be substituted into the approximate equations $P_X^*(r,s^0,s^1) - 1 \approx \Sigma_{n=1}^N w_{Xn}(r_n - 1)$ where X = L, P, F, T or W. The resulting equation will be exact for the Laspeyres index but, in general, it will not be exact for the remaining indexes. For the remaining indexes, the difference between $P_X^*(r,s^0,s^1) - 1$ and $\Sigma_{n=1}^N w_{Xn}(r_n - 1)$ can be labelled as a statistical discrepancy or the discrepancy could be distributed across the N contribution factors.

The above methodology for defining contribution factors is only one of many possible approaches. However, it does generate results that analysts will probably find suitable for their purposes.

We conclude this section by discussing the problems associated with deriving a decomposition of the rate of change of the Lowe index into explanatory factors involving rates of change for individual commodity prices.[58] This topic is of some interest since the European Union makes a great deal of use of the Lowe formula when computing its Harmonized Index of Consumer Prices

[56] Note that P_L^*, P_P^*, P_F^*, P_T^* and P_W^* all satisfy the identity test (60).
[57] If we approximate the geometric mean $(s_n^0 s_n^1)^{1/2}$ by the corresponding arithmetic mean, $(½)(s_n^0 + s_n^1)$, then it can be seen that $(½)(s_n^0 + s_n^1)/\Sigma_{i=1}^N (½)(s_i^0 + s_i^1) = (½)(s_n^0 + s_n^1)$ so that $w_{Wn} \approx (½)(s_n^0 + s_n^1) = w_{Fn} = w_{Tn}$. Thus the contribution factors for the Fisher, Törnqvist and Walsh indexes will all be approximately equal (and intuitively sensible).
[58] Our analysis is based on the work of Balk (2017), de Haan and Akem (2017) and Eurostat (2018; 180-183).

(HICP).[59] However, their Lowe index uses a combination of fixed base and chained indexes as will be seen below.

We consider the case where a statistical agency uses a fixed base Lowe index for 13 consecutive months in a year that starts in December; i.e., for the first 13 months of the index, December of, say 2018, is used as the base month. For these months, the following fixed base Lowe index is used at higher levels of aggregation:[60]

$$(68)\ P_{Lo}(p^0,p^t,q^{b0}) \equiv p^t\cdot q^{b0}/p^0\cdot q^{b0}\ ; \qquad t = 0,1,...,12$$
$$= \Sigma_{n=1}^N\ (p_n^t/p_n^0)p_n^0q_n^{b0}/p^0\cdot q^{b0}$$
$$= \Sigma_{n=1}^N\ s_n^{0b0}r_n$$

where p^0 is the monthly price vector for December 2018, $p^1,...,p^{12}$ are the relevant month price vectors for January-December of 2019, q^{b0} is a base year quantity vector for a prior year and the *price ratios* r_n^t and *hybrid shares* s_n^{0b0} are defined as follows:

$$(69)\ r_n^t \equiv p_n^t/p_n^0\ ;\ s_n^{0b0} \equiv p_n^0q_n^{b0}/p^0\cdot q^{b0} = (p_n^0/p_n^{b0})s_n^{b0}/\Sigma_{i=1}^N\ (p_i^0/p_i^{b0})s_i^{b0}\ ; \qquad n = 1,...,N$$

where $s_n^{b0} \equiv p_n^{b0}q_n^{b0}/p^0\cdot q^{b0}$ is the base year expenditure share for commodity n and p_n^{b0} and q_n^{b0} are the base year price and quantity for commodity n for n = 1,...,N. The second equation for the hybrid share s_n^{0b0} shows that it can be written using only the price ratios p_n^0/p_n^{b0} and the annual expenditure shares s_n^{b0} for base year 0.[61]

Since the shares s_n^{0b0} sum to 1, it can be seen that the Lowe index has the following exact decomposition into *commodity price change contribution factors*:

$$(70)\ P_{Lo}(p^0,p^t,q^{b0}) - 1 = \Sigma_{n=1}^N\ s_n^{0b0}\ (r_n^t - 1)\ ; \qquad t = 1,...,12.$$

However, typically analysts do not want to measure contributions to general inflation from December of the previous year to a particular month t in the subsequent year; they will be interested in month to month inflation in the subsequent year. It is not a problem to measure month to month inflation during the year subsequent to the base month. Since the Lowe index satisfies the circularity test for months t = 0, 1,...,12, the Lowe index going from month t to month t+1 is the following index:

$$(71)\ P_{Lo}(p^t,p^{t+1},q^{b0}) \equiv p^{t+1}\cdot q^{b0}/p^t\cdot q^{b0}\ ; \qquad t = 0,1,...,11$$
$$= \Sigma_{n=1}^N\ (p_n^{t+1}/p_n^t)p_n^tq_n^{b0}/p^t\cdot q^{b0}$$
$$= \Sigma_{n=1}^N\ s_n^{tb0}r_n^{t*}$$

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[59] See Chapter 8 of Eurostat (2018) for the details. This chapter was written by Bert Balk and Jens Mehrhoff. Other countries such as Australia and the UK use a similar annually chained Lowe index methodology. It should be noted that some member states of the European Union do not use an annually chained Lowe index at higher levels of aggregation; they use an annually chained Young index when constructing their HICP.

[60] All prices are assumed to be positive.

[61] It is also possible to show that the Lowe index $P_{Lo}(p^0,p^t,q^{b0}) \equiv p^t\cdot q^{b0}/p^0\cdot q^{b0}$ defined in terms of the annual basket vector q^{b0} can also be written as a function of *relative prices* and the *base year share vector*; i.e., we have $P_{Lo}(p^0,p^t,q^{b0}) = \Sigma_{n=1}^N\ (p_n^1/p_n^{b0})s_n^{b0}/\Sigma_{i=1}^N\ (p_i^1/p_i^{b0})s_i^{b0}$. It turns out that all of the formulae exhibited in this section can replace the use of q^{b0} by using an equivalent formula which uses the vector of base year expenditure shares. We will use the quantity vector q^{b0} in place of the base year share vector s^{b0} in order to simplify the formulae.

where the *short term price ratios* $r_n{}^*$ and *short term hybrid shares* $s_n{}^{tb0}$ are defined as follows:

(72) $r_n{}^{t*} \equiv p_n{}^{t+1}/p_n{}^t$; $s_n{}^{tb0} \equiv p_n{}^t q_n{}^{b0}/p^t \cdot q^{b0}$; $\qquad$ n = 1,...,N; t =1,...,11.

Since the shares $s_n{}^{tb0}$ sum to 1, it can be seen that the short term Lowe index has the following exact decomposition into *monthly commodity price change contribution factors*:

(73) $P_{Lo}(p^t,p^{t+1},q^{b0}) - 1 = \Sigma_{n=1}{}^N s_n{}^{tb0} (r_n{}^{t*} - 1)$; $\qquad$ t = 1,...,11.

Thus the problem of defining monthly contribution factors for the Lowe index for a single year is solved using the above framework. However, now suppose that the statistical agency does not use the base year quantity weights q^{b0} beyond one year; the annual weights are changed each year. The above algebra describes how the index is constructed for the months of say December 2018 through to December 2019. In December of 2019, a new annual quantity vector is introduced, say q^{b1}. The Lowe index for December 2018 is $P^{12} \equiv P_{Lo}(p^0,p^{12},q^{b0})$. The Lowe index using December 2018 as the reference price base for the next 12 months uses the new annual quantity vector q^{b1}. This new Lowe index is multiplied by the index level for December 2019 to give the overall index level P^t relative to December 2018 defined as follows:[62]

(74) $P^t \equiv P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^{12},p^t,q^{b1})$ $\qquad$ t = 13,...,24
$\quad = [p^{12} \cdot q^{b0}/p^0 \cdot q^{b0}][p^t \cdot q^{b1}/p^{12} \cdot q^{b1}]$
$\quad = [p^t \cdot q^{b1}/p^0 \cdot q^{b0}]/[p^{12} \cdot q^{b1}/p^{12} \cdot q^{b0}]$.

The first term in the last equality in (74) shows that the prices of month t (which is a month in 2020) are compared to the prices in month 0 (December of 2018) by the index $p^t \cdot q^{b1}/p^0 \cdot q^{b0}$. However the annual baskets, q^{b0} and q^{b1}, are not held constant in this comparison so this index is divided by a quantity index that compares the annual quantity vector q^{b1} to the prior annual quantity vector q^{b0}, using the price weights p^{12}, which are the monthly price weights for December 2019.[63]

Since the right hand side of (74) is a rather complicated function of p^0, p^1, p^{12}, q^{b0} and q^{b1}, it is difficult to develop a straightforward contributions to percentage change decomposition for this index. Many analysts will be interested in year over year contributions to overall percentage change. In this case, the price level in month 12 + t, P^{12+t}, is compared to the price level in month t of the base year, which is P^t, for t = 1,...,12. Using (68) and (74), this ratio is equal to:[64]

(75) $P^{12+t}/P^t = P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^{12},p^{12+t},q^{b1})/P_{Lo}(p^0,p^t,q^{b0})$; $\qquad$ t = 1,...,12
$\quad = [p^{12} \cdot q^{b0}/p^0 \cdot q^{b0}][p^{12+t} \cdot q^{b1}/p^{12} \cdot q^{b1}]/[p^t \cdot q^{b0}/p^0 \cdot q^{b0}]$
$\quad = [p^{12+t} \cdot q^{b1}/p^t \cdot q^{b0}][p^{12} \cdot q^{b0}/p^{12} \cdot q^{b1}]$.

[62] Note that the months have been numbered in a consecutive manner. Thus month 0 is December 2018, month 1 is January 2019,...,month 12 is December 2019, month 13 is January 2020,...., and month 24 is December 2020.

[63] Note that the index defined by (74) does not satisfy the identity test; i.e., it is not necessarily the case that $P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^t,p^{12},q^{b1}) = 1$ if $p^t = p^0$; see Balk (2017; 9). Of course, if $q^{b1} = q^{b0}$, then the identity test will be satisfied.

[64] This analysis follows that of Balk (2017; 8). Balk rearranged the terms in the last equality of (75) to give the following decomposition: $P^{12+t}/P^t = [p^{12+t} \cdot q^{b1}/p^{12} \cdot q^{b1}][p^{12} \cdot q^{b0}/p^t \cdot q^{b0}]$. Thus Balk noted that the year over year annually chained Lowe index is a product of two Lowe indexes where the first index uses quantity weights q^{b1} and the second uses the quantity weights q^{b0}.

Again, it is not straightforward to rewrite (75) as a function of the year over year price ratios, p_n^{12+t}/p_n^t for t = 1,...,12 and expenditure shares. However, it is possible to rewrite (75) in the following two alternative forms for t = 1,...,12:

$$(76)\ P^{12+t}/P^t = \kappa_{b0}[p^{12+t}\cdot q^{b0}/p^t\cdot q^{b0}]$$
$$= \kappa_{b0}[\Sigma_{n=1}^N s_n^{b0} (p_n^{12+t}/p_n^t)];$$
$$(77)\ P^{12+t}/P^t = \kappa_{b1}[p^{12+t}\cdot q^{b1}/p^t\cdot q^{b1}]$$
$$= \kappa_{b1}[\Sigma_{n=1}^N s_n^{b1} (p_n^{12+t}/p_n^t)]$$

where κ_{b0} and κ_{b1} are defined by (78) and the hybrid shares s_n^{b0} and s_n^{b1} are defined by (79):

$$(78)\ \kappa_{b0} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^{12+t}\cdot q^{b1}/p^{12+t}\cdot q^{b0}]\ ;\ \kappa_{b1} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^t\cdot q^{b1}/p^t\cdot q^{b0}]\ ;$$
$$(79)\ s_n^{b0} \equiv p_n^t q_n^{b0}/p^t\cdot q^{b0};\qquad s_n^{b1} \equiv p_n^t q_n^{b1}/p^t\cdot q^{b1}\ ;\qquad n = 1,...,N.$$

Note that indexes of the form $p\cdot q^{b1}/p\cdot q^{b0} \equiv Q_{Lo}(q^{b0},q^{b1},p)$ where p is a vector of reference prices are *Lowe type quantity indexes* which indicate the effects of the change in annual quantities going from q^{b0} to q^{b1}, holding prices fixed at p. It can be seen that using the definition of the Lowe quantity index, κ_{b0} and κ_{b1} can be written as follows:

$$(80)\ \kappa_{b0} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^{12+t}\cdot q^{b1}/p^{12+t}\cdot q^{b0}] = Q_{Lo}(q^{b0},q^{b1},p^{12+t})/Q_{Lo}(q^{b0},q^{b1},p^{12});$$
$$(81)\ \kappa_{b1} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^t\cdot q^{b1}/p^t\cdot q^{b0}] = Q_{Lo}(q^{b0},q^{b1},p^t)/Q_{Lo}(q^{b0},q^{b1},p^{12}).$$

Looking at (80) and (81), it can be seen that κ_{b0} and κ_{b1} represent the effects of changes in the annual quantity weights and it is likely that κ_{b0} and κ_{b1} will be close to one.

Using (76), we have the following exact decomposition of the year over year percentage change in the overall annually chained Lowe index:[65]

$$(82)\ [P^{12+t}/P^t] - 1 = \kappa_{b0}[\Sigma_{n=1}^N s_n^{b0}(p_n^{12+t}/p_n^t)] - 1\qquad t = 1,...,12$$
$$= \kappa_{b0}[\Sigma_{n=1}^N s_n^{b0}(p_n^{12+t}/p_n^t)] - \kappa_{b0} + \kappa_{b0} - 1$$
$$= \Sigma_{n=1}^N \kappa_{b0}s_n^{b0}[(p_n^{12+t}/p_n^t) - 1] + [\kappa_{b0} - 1].$$

Thus the year over year percentage change in the annually chained Lowe index for month t is no longer a *share weighted average* of the commodity price annual rates of change, $(p_n^{12+t}/p_n^t) - 1$; it is expressed as a *weighted sum* of the price changes $(p_n^{12+t}/p_n^t) - 1$ (with weight $\kappa_{b0}s_n^{b0}$ for commodity n) plus a term that reflects the changes in the annual quantity weights, which is $[\kappa_{b0} - 1]$. Of course, if κ_{b0} is equal to 1, then the quantity weights change term vanishes and (82) becomes the usual share weighted decomposition.

Using (77) instead of (76) leads to the following alternative exact decomposition of the year over year percentage change in the overall annually chained Lowe index:

$$(83)\ [P^{12+t}/P^t] - 1 = \kappa_{b1}[\Sigma_{n=1}^N s_n^{b1} (p_n^{12+t}/p_n^t)] - 1\qquad t = 1,...,12$$
$$= \kappa_{b1}[\Sigma_{n=1}^N s_n^{b1}(p_n^{12+t}/p_n^t)] - \kappa_{b1} + \kappa_{b1} - 1$$
$$= \Sigma_{n=1}^N \kappa_{b1}s_n^{b1}[(p_n^{12+t}/p_n^t) - 1] + [\kappa_{b1} - 1].$$

[65] This type of decomposition where there is a *separate term* for the effects of weight changes follows the methodological approach explained by de Haan and Akem (2017). Their approach to year over year contributions analysis was implemented by the Australian Bureau of Statistics (2017; 7).

If κ_{b1} equals one, then (83) collapses down to a traditional share weighted decomposition.

Since the decompositions defined by (82) and (83) are equally plausible, it is best to combine them into the following exact decomposition for t = 1,...,12:

$$(84)\ [P^{12+t}/P^{t}] - 1 = \Sigma_{n=1}^{N} (½)(\kappa_{b0}s_n^{b0} + \kappa_{b1}s_n^{b1})[(p_n^{12+t}/p_n^{t}) - 1] + [(½)(\kappa_{b0} + \kappa_{b1}) - 1].$$

The approach taken by Eurostat and the OECD to provide a decomposition of $[P^{12+t}/P^{t}] - 1$ is due to Ribe (1999) and it can be explained as follows. Using equation (75), we have

$$\begin{aligned}(85)\ P^{12+t}/P^{t} - 1 &= [p^{12+t}\cdot q^{b1}/p^{t}\cdot q^{b0}][p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}] - 1 \qquad t = 1,...,12 \\ &= [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}][p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1 \\ &= [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}]\{[p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1\} + [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}] - 1 \\ &= P_{Lo}(p^{t},p^{12},q^{b0})\{\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]\} + \{\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^{t}) - 1]\}\end{aligned}$$

where the hybrid shares s_n^{b0} were defined earlier in definitions (79) and the new hybrid shares s_n^{b1*} are defined as follows:

$$(86)\ s_n^{b1*} \equiv p_n^{12}q_n^{b1}/p^{12}\cdot q^{b1}\ ; \qquad n = 1,...,N.$$

The first term on the right hand side of the last equation in (85) is regarded as a "this year" term that looks at the contribution of price change from month 12 to month 12 + t, while the second term, $\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^{t}) - 1]$, is regarded as a "last year" contribution term that looks at the price change from month t in the first year to month 12 in the first year.

Balk commented on the decomposition defined by (85) as follows:

> "However, by looking at the structure of the right-hand side [of (85)] it becomes clear that this decomposition is not completely satisfactory. Though the second factor between brackets can be interpreted as previous year's contribution, and the first factor between brackets likewise as current year's contribution (and both factors can be decomposed commodity-wise), this first factor is multiplied by previous year's price change. Thus there seems to be a whiff of double-counting here." Bert M. Balk (2017; 9)

Thus Balk noted that the Lowe index $P_{Lo}(p^{t},p^{12},q^{b0})$ precedes the first price decomposition term, $\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]$, and this Lowe index involves the overall amount of inflation going from month t in the first year to December of the first year and this amount of overall inflation estimate augments the commodity specific contributions going from December of the first year to month t of the second year. The decomposition defined by (84) seems to be conceptually cleaner with year over year contributions to overall year over year inflation listed in single terms (rather than as a sum of two terms) plus a final term that measures the overall contribution made by the changing annual baskets.[66]

It is possible to obtain an alternative decomposition to (85) that is equally plausible.[67] Again using equation (75), we have:

$$(87)\ [P^{12+t}/P^{t}] - 1 = [p^{12+t}\cdot q^{b1}/p^{t}\cdot q^{b0}][p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}] - 1 \qquad t = 1,...,12$$

[66] De Haan and Akem's (2017) decomposition is similar in structure to (84) except that their decomposition of overall inflation is not an exact one.

[67] The analysis which follows is due to Jens Mehrhoff.

$$= [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}][p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}] - 1$$
$$= [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}]\{[p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}] - 1\} + [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1$$
$$= P_{Lo}(p^{12},p^{12+t},q^{b1})\{\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^{t}) - 1]\} + \{\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]\}$$

where the hybrid shares s_n^{b0} are defined by (79) and the hybrid shares s_n^{b1*} are defined by (86).

When two equally plausible estimates for the same thing are available and a single estimate is required, it is best to take an evenly weighted average of the two estimates to form a final estimate. Thus taking the arithmetic mean of the two estimates defined by (85) and (87) leads to the following *Mehrhoff decomposition* of year over year price change into explanatory components:

$$(88)\ [P^{12+t}/P^{t}] - 1 = \Sigma_{n=1}^{N} (½)s_n^{b0}[1 + P_{Lo}(p^{12},p^{12+t},q^{b1})][(p_n^{12}/p_n^{t}) - 1] \qquad t = 1,...,12$$
$$+ \Sigma_{n=1}^{N} (½)s_n^{b1*}[1 + P_{Lo}(p^{t},p^{12},q^{b0})][(p_n^{12+t}/p_n^{12}) - 1].$$

Of course, there are many other decompositions that have been suggested in the literature.[68] As was indicated earlier, there is no unambiguous "best" solution to this decomposition problem.

Appendix: Proof of Propositions

Proof of Proposition 1: Using the positivity test T1, rewrite the circularity test (28) in the following form:

$$(A1)\ \ P(p^1,p^2,q^1,q^2) = P(p^0,p^2,q^0,q^2)/P(p^0,p^1,q^0,q^1).$$

Now hold p^0 and q^0 constant at some fixed values, say $p^* >> 0_N$ and $q^* >> 0_N$ and define the function f(p,q) as follows:

$$(A2)\ \ f(p,q) \equiv P(p^*,p,q^*,q) > 0 \qquad \text{for all } p>>0_N \text{ and } q >> 0_N$$

where the positivity of f(p,q) follows from T1. Substituting definition (A2) back into (A1) gives us the following representation for $P(p^1,p^2,q^1,q^2)$:

$$(A3)\ \ P(p^1,p^2,q^1,q^2) = f(p^2,q^2)/f(p^1,q^1).$$

Now let $p^1 = p^2 = p$ in (A3) and apply the identity test T3 to the resulting equation. We obtain:

$$(A4)\ \ 1 = P(p,p,q^1,q^2) = f(p,q^2)/f(p,q^1)\ ; \qquad p >> 0_N;\ q^1 >> 0_N;\ q^2 >> 0_N.$$

Equations (A4) imply that $f(p,q^1) = f(p,q^2)$ for all $p >> 0_N$; $q^1 >> 0_N$; $q^2 >> 0_N$ which in turn implies that f(p,q) does not depend on q; i.e., we have

$$(A5)\ f(p,q) = f(p,q^*) \text{ for all } p >> 0_N;\ q >> 0_N.$$

Define the function c(p) for all $p >> 0_N$ as

$$(A6)\ \ c(p) \equiv f(p,q^*)$$
$$= P(p^*,p,q^*,q^*)$$

[68] See Walshots (2016), Balk (2017), de Haan and Akem (2017), OECD (2018) and Chapter 8 in Eurostat (2018).

Substitute (A5) and (A6) back into (A3) and we obtain the following representation for the index number formula, $P(p^1,p^2,q^1,q^2)$:

(A7) $P(p^1,p^2,q^1,q^2) = c(p^2)/c(p^1)$.

Now apply the commensurability test, T10, to the P that is defined by (A7) where we set $\alpha_n = (p_n^0)^{-1}$ for n = 1,...,N. Using the representation for P given by (61), we find that c must satisfy the following functional equation:

(A8) $c(p^1)/c(p^0) = c(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/c(1_N)$; $\qquad p^0 >> 0_N;\ p^1 >> 0_N$.

Define h(p) as follows:

(A9) $h(p) \equiv c(p)/c(1_N) > 0$; $\qquad p >> 0_N$

where the positivity of h follows from the positivity of c. Using definition (A9), we have:

$$
\begin{aligned}
\text{(A10)}\quad h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0) &= c(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/c(1_N) && p^0 >> 0_N;\ p^1 >> 0_N \\
&= c(p^1)/c(p^0) && \text{using (A8)} \\
&= [c(p^1)/c(1_N)]/[c(p^0)/c(1_N)] && \text{using T1} \\
&= h(p^1)/h(p^0) && \text{using (A9) twice.}
\end{aligned}
$$

Thus h must satisfy the following functional equation:

(A11) $h(p^0)h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_1^1/p_1^0) = h(p^1)$; $\qquad p^0 >> 0_N;\ p^1 >> 0_N$.

Define the vector x as the vector p^0 and the vector y as $p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0$. Hence the product of the nth components of x and y is equal to the nth component of the vector p^1 and it can be seen that the functional equation (A11) is equivalent to the following functional equation:

(A12) $h(x_1y_1,x_2y_2,...,x_Ny_N) = h(x_1,x_2,...,x_N)h(y_1,y_2,...,y_N)$; $\qquad x >> 0_N;\ y >> 0_N$.

Equation (A12) becomes the following equation if we allow x_1 and y_1 to vary freely but fix all x_i and y_i at 1 for i = 2,3,...,N:

(A13) $h(x_1y_1,1,...,1) = h(x_1,1,...,1)h(y_1,1,...,1)$; $\qquad x_1 > 0;\ y_1 > 0$.

But (A13) is an example of *Cauchy's* (1821) *fourth functional equation*. Using the T1 (positivity) and T2 (continuity) properties of P, which carry over to h, we see that the solution to (A13) is:

(A14) $h(x_1,1,...,1) = x_1^{c(1)}$

where c(1) is an arbitrary constant. In a similar fashion, (A12) becomes the following equation if we allow x_2 and y_2 to vary freely but fix all other x_i and y_i at 1:

(A15) $h(1,x_2y_2,1,...,1) = h(1,x_2,1,...,1)h(1,y_2,1,...,1)$; $\qquad x_2 > 0;\ y_2 > 0$.

The solution to (A15) is:

(A16) $h(1,x_2,1,...,1) = x_2^{c(2)}$

where c(2) is an arbitrary constant. In a similar fashion, we find that

(A17) $h(1,1,x_3,1,...,1) = x_3^{c(3)}$; ... ; $h(1,1,...,1,x_N) = x_N^{c(N)}$

where the c(i) are arbitrary constants. Using (66) repeatedly, we can show:

(A18)
$$\begin{aligned} h(x_1,x_2,...,x_N) &= h(x_1,1,...,1)h(1,x_2,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)h(1,1,1,x_4,...,x_N) \\ &\;\;\; ... \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)...h(1,1,1,...,1,x_N) \\ &= \textstyle\prod_{i=1}^N x_i^{c(i)} \qquad \text{using (A14), (A16) and (17).} \end{aligned}$$

Thus we have determined the functional form for the function h. Now use (A9) to determine the function c(p) in terms of h(p):

(A19)
$$\begin{aligned} c(p) &= c(1_N)h(p) \\ &= c(1_N) \textstyle\prod_{i=1}^N p_i^{c(i)}. \end{aligned}$$

Using (A7), we can express P in terms of c as follows:

(A20)
$$\begin{aligned} P(p^0,p^1,q^0,q^1) &= c(p^1)/c(p^0) \\ &= c(1_N) \textstyle\prod_{i=1}^N (p_i^1)^{c(i)}/c(1_N) \prod_{i=1}^N (p_i^0)^{c(i)} \qquad \text{using (A19)} \\ &= \textstyle\prod_{i=1}^N (p_i^1/p_i^0)^{c(i)}. \end{aligned}$$

Now apply test T5, proportionality in current prices, to the P defined by (A20). It is easy to see that this test implies that the constants c(i) must sum to 1.

Finally, apply test T17, monotonicity in current prices, to conclude that the constants c(i) must be positive. Hence we can set the c(i) equal to the α_i and we have proved the Proposition.

It should be noted that Konüs and Byushgens (1926) and Frisch (1930) provided alternative proofs for this result, assuming differentiability of the price index function. They used solutions to partial differential equations in place of Cauchy's fourth fundamental functional equation.

Proof of Proposition 2: Define $r_i \equiv p_i^1/p_i^0$ for i = 1,…,N. Using T1, T9 and (41), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7 and (41):

(A21) $P(p^0,p^1,v^0,v^1) = P^*(r,s^0,s^1)$

where s^t is the period t expenditure share vector for t = 0,1.

Let $x \equiv (x_1,…,x_N)$ and $y \equiv (y_1,…,y_N)$ be strictly positive vectors. The transitivity test T11 and (A21) imply that the function P^* has the following property:

(A22) $P^*(x,s^0,s^1)P^*(y,s^0,s^1) = P^*(x_1y_1,…,x_Ny_N,s^0,s^1)$.

Using T1, $P^*(r,s^0,s^1) > 0$ and using T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

(A23) $P^*(1_N,s^0,s^1) = 1$

where 1_N is a vector of ones of dimension N. Using a result due to Eichhorn (1978; 66), it can be seen that these properties of P^* are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for i = 1,…,N such that P^* has the following representation:

(A24) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \alpha_i(s^0,s^1)\ln r_i$.

The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

$$\begin{aligned}
\text{(A25) } \ln P^*(\lambda r,s^0,s^1) &= \ln\lambda + \ln P^*(r,s^0,s^1) \\
&= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda r_i && \text{using (A24)} \\
&= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda + \sum_{i=1}^N \alpha_i(s^0,s^1)\ln r_i \\
&= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda + \ln P^*(r,s^0,s^1) && \text{using (A24) again.}
\end{aligned}$$

Equating the right hand sides of the first and last lines in (A25) shows that the functions $\alpha_i(s^0,s^1)$ must satisfy the following restriction:

(A26) $\sum_{i=1}^N \alpha_i(s^0,s^1) = 1$

for all strictly positive vectors s^0 and s^1.

Using the weighting test T16 and the commodity reversal test T8, equations (43) hold. Equations (43) combined with the commensurability test T9 implies that P^* satisfies the following equations:

(A27) $P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = f(1,s_i^0,s_i^1)$; $i = 1,\ldots,N$

for all $r_i > 0$ where f is the function defined in test T16.

Substitute equations (A27) into equations (A24) in order to obtain the following system of equations:

(A28) $\ln P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = \ln f(1,s_i^0,s_i^1) = \alpha_i(s^0,s^1)\ln r_i$; $i = 1,\ldots,N$.

But equation i in (A28) implies that the positive continuous function of 2N variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except s_i^0 and s_i^1 and this property holds for each i. Thus each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\beta_i(s_i^0,s_i^1)$ for i = 1,…,N.[69] Now replace the $\alpha_i(s^0,s^1)$ in equation (A24) by the $\beta_i(s_i^0,s_i^1)$ for i = 1,…,N and the following representation for P^* is obtained:

(A29) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \beta_i(s_i^0,s_i^1)\ln r_i$

Equations (A26) imply that the functions $\beta_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

[69] More explicitly, $\beta_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ and so on. That is, in defining $\beta_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ is used where all components of the vectors s^0 and s^1 except the first are set equal to an arbitrary positive number like 1.

(A30) $\sum_{n=1}^{N} s_n^{0} = 1$; $\sum_{n=1}^{N} s_n^{1} = 1$ implies $\sum_{i=1}^{N} \beta_i(s_i^{0},s_i^{1}) = 1$.

Assume that the weighting test T17 holds and substitute equations (43) into (A29) in order to obtain the following equations:

(A31) $\beta_i(0,0) \ln [p_i^{1}/p_i^{0}] = 0$; $\qquad i = 1,\ldots,N$.

Since the p_i^{1} and p_i^{0} can be arbitrary positive numbers, it can be seen that (A31) implies

(A32) $\beta_i(0,0) = 0$; $\qquad i = 1,\ldots,N$.

Assume that the number of commodities N is equal to or greater than 3. Using (A10) and (A12), Theorem 2 in Aczél (1987; 8) can be applied and the following functional form for each of the $\beta_i(s_i^{0},s_i^{1})$ is obtained:

(A33) $\beta_i(s_i^{0},s_i^{1}) = \gamma s_i^{0} + (1-\gamma)s_i^{1}$; $\qquad i = 1,\ldots,N$

where γ is a positive number satisfying $0 < \gamma < 1$.

Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that γ must equal ½. Substituting this value for γ back into (A33) and then substituting those equations back into (A29), the functional form for P^* and hence P is determined as

(A34) $\ln P(p^0,p^1,v^0,v^1) = \ln P^*(r,s^0,s^1) = \sum_{n=1}^{N} (1/2)[s_n^{0} + s_n^{1}]\ln (p_n^{1}/p_n^{0})$.

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