The Difference Approach to Productivity Measurement and Exact Indicators

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Abstract

There are many decompositions of productivity growth for a production unit that rely on the ratio approach to index number theory. However, the business and accounting literatures tend to favour using differences rather than ratios. In this paper, three analogous decompositions for productivity growth in a difference approach to index number theory are obtained. The first approach uses the production unit's value added function in order to obtain a suitable decomposition. It relies on various first order approximations to this function, but the decomposition can be given an axiomatic interpretation. The second approach uses the cost constrained value added function and assumes that the reference technology for the production unit can be approximated by the free disposal conical hull of past observations of inputs used and outputs produced by the unit. The final approach uses a particular flexible functional form for the producer's value added function and provides an exact decomposition of normalized value added.

Keywords

Productivity measurement, index numbers, indicator functions, the Bennet indicator, flexible functional forms for value added functions, technical and allocative efficiency, nonparametric methods for production theory, measures of technical progress.

JEL Classification Numbers

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1. Introduction

Total Factor Productivity (TFP) growth is usually defined by economists as an output index divided by an input index. However, in the business and accounting literatures, there is more interest in measuring productivity growth in a difference framework.[2] Thus in the present paper, we will look at the value added produced by a production unit in two consecutive periods of time and we will attempt to find a decomposition of the value added difference into explanatory factors. One of these factors will be productivity growth measured in a difference framework.

We will develop three separate approaches to the value added decomposition problem in difference format. The first approach will be explained in sections 2 and 3. This approach relies on the assumption that observed production is always on the frontier of the production possibilities set and makes use of various first order approximations to the underlying value added functions. Approach 1 can be given an axiomatic interpretation which has some good properties. Section 4 notes a problem with the difference approach: nominal value added measured at two different points in time is measured in monetary units but the value of the monetary unit is not constant over time. Thus in section 4, we suggest that all prices be deflated by a suitable general index of inflation.

Section 5 no longer assumes that producers are necessarily on the frontiers of their production possibility sets. The analysis here makes use of cost constrained value added functions and it also assumes that the producer's period t production possibilities set can be approximated by the free disposal conical hull of past observations on outputs produced and inputs used by the unit. Thus Approach 2 is a nonparametric one, as is Approach 1.

Section 6 tries to develop a counterpart to the index number decompositions for value added growth that were obtained by Diewert and Morrison (1986) and Kohli (1990) that would be applicable in the difference approach to economic measurement (as opposed to the ratio approach that was used in these earlier studies). We succeed in providing a counterpart decomposition but in the end, it proves to be not very useful.

Section 7 concludes with some observations on the relative merits and demerits of the three approaches.

2. The First Order Approximation Approach

Let $y \equiv [y_1,...,y_M]$ denote an M dimensional vector of net outputs (if $y_m > 0$, then net output m is an output, if $y_m < 0$, then net output m is an intermediate input) and let $x \equiv [x_1,...,x_N] \geq 0_N$ denote a nonnegative N dimensional vector of primary inputs. We want to look at the productivity of a production unit that produces the M net outputs using the N primary inputs over periods t = 0, 1,...,T. We assume that the period t production possibilities set is a set of feasible combinations of net outputs and primary inputs

[2] See Grifell-Tatjé and Lovell (2015) for an extensive discussion of the difference approach to productivity measurement and its history.

denoted by the set S^t. For each t, we assume a constant returns to scale production possibilities set so that S^t is a cone.[3] Suppose the producer faces the strictly positive vector of net output prices $p^t \equiv [p_1^t,...,p_M^t] >> 0_M$ in period t and has the nonnegative vector of primary inputs $x^t \equiv [x_1^t,...,x_N^t]$ at its disposal. The maximum value added that the production unit can produce is $\Pi^t(p^t,x^t)$ defined as follows for t = 0, 1,...,T:

(1) $\Pi^t(p^t,x^t) \equiv \max_y \{p^t \cdot y : (y,x^t) \in S^t\}$

where $\Pi^t(p,x)$ is the producer's *period t value added function*.[4] Let $w^t \equiv [w_1^t,...,w_N^t] >> 0_N$ be the period t vector of positive primary input prices. We assume that observed value added is equal to observed primary input cost in each period:[5]

(2) $\Pi^t(p^t,x^t) \equiv p^t \cdot y^t = w^t \cdot x^t > 0$; t = 0,1,...,T.

In this section, we also assume that $\Pi^t(p^t,x^t)$ is differentiable with respect to its components in each period so that we have:

(3) $y^t = \nabla_p \Pi^t(p^t,x^t)$; t = 0,1,...,T; (Hotelling's (1932; 594) Lemma)
(4) $w^t = \nabla_x \Pi^t(p^t,x^t)$; t = 0,1,...,T; (Samuelson's (1953; 10) Lemma).[6]

We focus on the growth of value added going from period 0 to period 1. The growth analysis for other periods is entirely analogous. Assumptions (2) above plus simple algebra establishes the following *Laspeyres and Paasche type value added growth decompositions in ratio form*:

(5) $p^1 \cdot y^1/p^0 \cdot y^0 = [\Pi^1(p^1,x^1)/\Pi^1(p^0,x^1)][\Pi^1(p^0,x^1)/\Pi^0(p^0,x^1)][\Pi^0(p^0,x^1)/\Pi^0(p^0,x^0)]$;
(6) $p^1 \cdot y^1/p^0 \cdot y^0 = [\Pi^0(p^1,x^0)/\Pi^0(p^0,x^0)][\Pi^1(p^1,x^0)/\Pi^0(p^1,x^0)][\Pi^1(p^1,x^1)/\Pi^1(p^1,x^0)]$.

The terms $\Pi^t(p^1,x^t)/\Pi^t(p^0,x^t)$ for t = 0,1 are *value added price indexes*, the terms $\Pi^1(p^0,x^1)/\Pi^0(p^0,x^1)$ and $\Pi^1(p^1,x^0)/\Pi^0(p^1,x^0)$ are *measures of technical progress* and the terms $\Pi^t(p^t,x^1)/\Pi^t(p^t,x^0)$ for t = 0,1 are *input quantity indexes*. These ratio type decompositions have the following analogous *Laspeyres and Paasche type value added difference decompositions*:[7]

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[3] In addition to the cone property, we assume the weak regularity conditions P1-P7 on S^t that are listed in Diewert and Fox (2017; 277). Essentially, we assume that S^t is a nonempty closed cone which is subject to free disposal and $(0_M,0_N) \in S^t$ for each t. It is not necessary to assume that S^t is a convex set. However, the assumption of constant returns to scale in production is restrictive (but necessary for our analysis).
[4] For the properties of value added functions, see McFadden (1966) (1968), Gorman (1968) and Diewert (1973). In this section, we will assume that first order partial derivatives of $\Pi^t(p,x)$ at $p = p^t$ and $x = x^t$ exist.
[5] In empirical applications, there are two main methods for ensuring that the value of outputs equals the value of inputs: (i) introduce a fixed factor that absorbs any pure profits or losses or (ii) use a balancing rate of return in the user cost formula for durable inputs that will make the value of inputs equal to the value of outputs. For the early history of the first approach, see Grifell-Tatjé and Lovell (2015; 40) and for applications of the second approach, see Christensen and Jorgenson (1969) and Diewert and Fox (2018a).
[6] See also Diewert (1974; 140).
[7] Hicks (1941-2; 127-134) (1945-6; 72-73) was the first to see the analogy between index number theory and consumer surplus theory (a difference approach to welfare measurement) and he developed a first order

$$(7)\ p^1\cdot y^1 - p^0\cdot y^0 = [\Pi^1(p^1,x^1) - \Pi^1(p^0,x^1)] + [\Pi^1(p^0,x^1) - \Pi^0(p^0,x^1)] + [\Pi^0(p^0,x^1) - \Pi^0(p^0,x^0)];$$
$$(8)\ p^1\cdot y^1 - p^0\cdot y^0 = [\Pi^0(p^1,x^0) - \Pi^0(p^0,x^0)] + [\Pi^1(p^1,x^0) - \Pi^0(p^1,x^0)] + [\Pi^1(p^1,x^1) - \Pi^1(p^1,x^0)].$$

The terms $\Pi^t(p^1,x^t) - \Pi^t(p^0,x^t)$ are value added *indicators* of price change,[8] the terms $\Pi^1(p^0,x^1) - \Pi^0(p^0,x^1)$ and $\Pi^1(p^1,x^0) - \Pi^0(p^1,x^0)$ are measures of the absolute change in value added at constant net output prices and constant primary input quantities due to *technical progress* going from period 0 to 1 and the terms $\Pi^t(p^t,x^1) - \Pi^t(p^t,x^0)$ for t = 0,1 are *indicators* of input growth at constant prices and constant technology in difference terms. Our problem is to obtain empirically observable estimates for the three sets of terms on the right hand sides of (7) and (8).

We will use assumptions (2)-(4) in order to form first order Taylor series approximations to the various unobservable value added terms of the form $\Pi^r(p^s,x^t)$ for r,s and t equal to 0 or 1. Thus we can derive the following first order approximations to the unobservable terms on the right hand sides of the decompositions defined by (7) and (8):

$$\begin{aligned}(9)\ \Pi^1(p^1,x^1) - \Pi^1(p^0,x^1) &\approx p^1\cdot y^1 - [\Pi^1(p^1,x^1) + \nabla_p\Pi^1(p^1,x^1)\cdot(p^0-p^1)] \\ &= p^1\cdot y^1 - [p^1\cdot y^1 + y^1\cdot(p^0-p^1)] && \text{using (2) and (3)} \\ &= y^1\cdot(p^1-p^0).\end{aligned}$$

$$\begin{aligned}(10)\ &\Pi^1(p^0,x^1) - \Pi^0(p^0,x^1) \\ &\approx [\Pi^1(p^1,x^1) + \nabla_p\Pi^1(p^1,x^1)\cdot(p^0-p^1)] - [\Pi^0(p^0,x^0) + \nabla_x\Pi^0(p^0,x^0)\cdot(x^1 - x^0)] \\ &= [p^1\cdot y^1 + y^1\cdot(p^0 - p^1)] - [p^0\cdot y^0 + w^0\cdot(x^1 - x^0)] && \text{using (2)-(4)} \\ &= p^0\cdot y^1 - w^0\cdot x^1.\end{aligned}$$

$$\begin{aligned}(11)\ \Pi^0(p^0,x^1) - \Pi^0(p^0,x^0) &\approx [\Pi^0(p^0,x^0) + \nabla_x\Pi^0(p^0,x^0)\cdot(x^1 - x^0)] - p^0\cdot y^0 && \text{using (2)} \\ &= [p^0\cdot y^0 + w^0\cdot(x^1 - x^0)] - p^0\cdot y^0 && \text{using (4)} \\ &= w^0\cdot(x^1 - x^0).\end{aligned}$$

Thus we have derived observable approximations to each of the unobservable components in the decompositions (7) and (8).

Substituting (9)-(11) into the decomposition (7) gives us the following approximate decomposition:[9]

$$(12)\ p^1\cdot y^1 - p^0\cdot y^0 \approx [y^1\cdot(p^1 - p^0)] + [p^0\cdot y^1 - w^0\cdot x^1] + [w^0\cdot(x^1 - x^0)] = p^1\cdot y^1 - p^0\cdot y^0.$$

Taylor series approximation method to obtain empirical counterparts to his theoretical difference measures of quantity and price change. Thus we are simply adapting his method to the producer context. See Diewert and Mizobuchi (2009; 367) for the early history of the contributions of Hicks to indicator theory in the consumer context.

[8] Diewert (1992; 556) introduced the term indicator to distinguish the difference concept from the usual ratio concept that is applied in index number theory. Diewert (2005; 317) also applied the indicator terminology in the context of measuring profit change over consecutive periods.

[9] We used $w^0\cdot x^0 = p^0\cdot y^0$ to derive the last equality in (12).

Similar computations give us the following first order Taylor series approximations to the three terms on the right hand side of decomposition (8):

(13) $\Pi^0(p^1,x^0) - \Pi^0(p^0,x^0) \approx y^0\cdot(p^1 - p^0)$;
(14) $\Pi^1(p^1,x^0) - \Pi^0(p^1,x^0) \approx -[p^1\cdot y^0 - w^1\cdot x^0]$;
(15) $\Pi^1(p^1,x^1) - \Pi^1(p^1,x^0) \approx w^1\cdot(x^1 - x^0)$.

Substituting (13)-(15) into (8) gives us the following approximate decomposition, expressed in terms of observable data:[10]

(16) $p^1\cdot y^1 - p^0\cdot y^0 \approx [y^0\cdot(p^1 - p^0)] - [p^1\cdot y^0 - w^1\cdot x^0] + [w^1\cdot(x^1 - x^0)] = p^1\cdot y^1 - p^0\cdot y^0$.

Now take the arithmetic average of the two approximate decompositions (12) and (16) and we obtain the following Bennet (1920) type approximate decomposition:

$$(17)\ p^1\cdot y^1 - p^0\cdot y^0 \approx (1/2)(y^0 + y^1)\cdot(p^1 - p^0) + (1/2)[(p^0\cdot y^1 - w^0\cdot x^1) - (p^1\cdot y^0 - w^1\cdot x^0)] + (1/2)(w^0 + w^1)\cdot(x^1 - x^0) = p^1\cdot y^1 - p^0\cdot y^0.$$

This last equality follows by simply adding up the terms. Define the *Bennet indicator of technical progress*, $B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1)$, as the middle term in the above decomposition:[11]

(18) $B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1) \equiv (1/2)[(p^0\cdot y^1 - w^0\cdot x^1) - (p^1\cdot y^0 - w^1\cdot x^0)]$.

The last equality in (17) shows that the approximate decomposition (17) is in fact an exact one in the sense that the sum of the right hand side terms equals the value added difference between the two periods. The first term on the right hand side of (17), $(1/2)(y^0 + y^1)\cdot(p^1 - p^0)$, is the *Bennet indicator of value added price change*, the middle term, $(1/2)[(p^0\cdot y^1 - w^0\cdot x^1) - (p^1\cdot y^0 - w^1\cdot x^0)]$, is an *indicator of technical progress between periods 0 and 1* and the last term, $(1/2)(w^0 + w^1)\cdot(x^1 - x^0)$, is the *Bennet indicator of input quantity change*. Note that the Bennet indicator of technical progress turns out to equal the arithmetic average of the hypothetical profit that the net output vector of period 1 would make if evaluated at the prices of period 0 and the negative of the hypothetical loss that the net output vector of period 0 would make if evaluated at the prices of period 1. This shows that there is a strong connection between measures of technical progress and of profitability.[12]

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[10] We used $-p^0\cdot y^0 = -w^0\cdot x^0$ to derive the last equality in (16).

[11] Kurosawa (1975) recognized $p^0\cdot y^1 - w^0\cdot x^1$ as a measure of productivity growth (or technical progress); see also Grifell-Tatjé and Lovell (2015; 177-185) for a discussion of this measure and related measures.

[12] See Grifell-Tatjé and Lovell (2015) for much more material on the relationships of profitability measures with measures of productivity growth.

The classic *Bennet decomposition of value added change* into price change and quantity change components going from period 0 to 1 is the following one:

(19) $p^1\cdot y^1 - p^0\cdot y^0 = (1/2)(y^0 + y^1)\cdot(p^1 - p^0) + (1/2)(p^0 + p^1)\cdot(y^1 - y^0)$

where the *Bennet indicator of value added quantity change* is defined as $(1/2)(p^0 + p^1)\cdot(y^1 - y^0)$. Substituting (19) into (17) and using definition (18) leads to the following two alternative expressions for the *Bennet indicator of technical progress*:

$$(20)\ B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1) = (1/2)(p^0 + p^1)\cdot(y^1 - y^0) - (1/2)(w^0 + w^1)\cdot(x^1 - x^0)$$
$$= (1/2)(p^0 + p^1)\cdot y^1 - (1/2)(w^0 + w^1)\cdot x^1 - [(1/2)(p^0 + p^1)\cdot y^0 - (1/2)(w^0 + w^1)\cdot x^0]$$

The first equality in (20) shows that the Bennet indicator of technical progress is also equal to the Bennet indicator of value added quantity change less the Bennet indicator of input quantity change. The second equality in (20) shows that the Bennet indicator of technical progress is equal to the hypothetical profitability of the overall period 1 net output vector, $[y^1, -x^1]$, evaluated at the average prices for period 0 and 1 net outputs, $(1/2)[p^0+p^1,w^0+w^1]$, less the hypothetical profitability of the overall period 0 net output vector, $[y^0, -x^0]$, evaluated at the same average prices for period 0 and 1 net outputs.

It is possible to obtain a fourth (dual) expression[13] for the Bennet indicator of technical progress. We know that value added change has the Bennet decomposition defined by (19) above and primary input cost change has the Bennet decomposition defined by (21) below:

(21) $w^1\cdot x^1 - w^0\cdot x^0 = (1/2)(w^0 + w^1)\cdot(x^1 - x^0) + (1/2)(x^0 + x^1)\cdot(w^1 - w^0)$.

Using (20), we have:

$$(22)\ B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1) = (1/2)(p^0 + p^1)\cdot(y^1 - y^0) - (1/2)(w^0 + w^1)\cdot(x^1 - x^0)$$
$$= -[p^1\cdot y^1-p^0\cdot y^0] + (1/2)(p^0+p^1)\cdot(y^1-y^0) + [w^1\cdot x^1-w^0\cdot x^0] - (1/2)(w^0+w^1)\cdot(x^1-x^0) \quad \text{using (2) for } t = 0,1$$
$$= -(1/2)(y^0+y^1)\cdot(p^1-p^0) + (1/2)(x^0+x^1)\cdot(w^1-w^0) \quad \text{using (19) and (21)}$$
$$= -\Sigma_{m=1}^{M} (1/2)(y_m^1+y_m^0)(p_m^1-p_m^0) + \Sigma_{n=1}^{N} (1/2)(x_n^1+x_n^0)(w_n^1-w_n^0).$$

Thus the empirical measure of technical progress $B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1)$ defined by (18) is also equal to the empirical Bennet measure of input price change $(1/2)(x^0+x^1)\cdot(w^1-w^0)$ less the Bennet measure of output price change, $(1/2)(y^0+y^1)\cdot(p^1-p^0)$.[14]

[13] The dual approach to productivity measurement dates back to Siegel (1952). See also Jorgenson and Griliches (1967) and Grifell-Tatjé and Lovell (2015; 103-109) for more on the early history of this approach.

[14] Jorgenson and Griliches (1967; 252) derived the ratio version of this price dual to the quantity indicator of technical progress.

We then have four alternative interpretations for the Bennet indicator of technical progress: the first one which flows from the original definition (18), two more interpretations which flow from the two equalities in (20) and the final dual interpretation defined by (22). All four interpretations are fairly simple and intuitively plausible.

The empirical decomposition of productivity growth defined by (22) is a useful if one wishes to allocate aggregate productivity growth to purchasers of the production unit's outputs and to suppliers of primary inputs to the production unit. Thus the benefits of productivity growth flow through to (potentially) lower net output prices (this effect is captured by the terms $-\Sigma_{m=1}^{M}(1/2)(y_m^1+y_m^0)(p_m^1-p_m^0)$ on the right hand side of (22)) and to higher primary input prices (this effect is captured by the terms $\Sigma_{n=1}^{N}(1/2)(x_n^1+x_n^0)(w_n^1-w_n^0)$ on the right hand side of (22)).[15]

3. Decomposing the Theoretical Indicators of Overall Output Price and Input Quantity Change into Individual Price and Quantity Indicators

Recall that the decomposition of value added growth defined by (7) had the overall output price change term $\Pi^1(p^1,x^1) - \Pi^1(p^0,x^1)$ on the right hand side of the equation. It is useful to decompose this Paasche type *overall* measure of output price change into *separate* output price change contributions.[16] This task can be accomplished if we make use of the following decomposition of $\Pi^1(p^1,x^1) - \Pi^1(p^0,x^1)$:

$$(23)\ \Pi^1(p^1,x^1) - \Pi^1(p^0,x^1) = \Pi^1(p^1,x^1) - \Pi^1(p_1^0,p_2^1,\ldots,p_M^1,x^1)$$
$$+ \Pi^1(p_1^0,p_2^1,\ldots,p_M^1,x^1) - \Pi^1(p_1^0,p_2^0,p_3^1,\ldots,p_M^1,x^1)$$
$$+ \Pi^1(p_1^0,p_2^0,p_3^1,\ldots,p_M^1,x^1) - \Pi^1(p_1^0,p_2^0,p_3^0,p_4^1\ldots,p_M^1,x^1)$$
$$+ \ldots$$
$$+ \Pi^1(p_1^0,\ldots,p_{M-1}^0,p_M^1,x^1) - \Pi^1(p^0,x^1).$$

Thus the right hand side of (23) consists of M differences in $\Pi^1(p,x^1)$ where each difference changes only one component of the p vector. We will approximate these terms by taking first order Taylor series approximations to the $\Pi^1(p,x^1)$ around the point $p = p^1$. Thus the first order approximation to $\Pi^1(p_1^0,p_2^1,\ldots,p_M^1,x^1)$ is the following one:

$$(24)\ \Pi^1(p_1^0,p_2^1,\ldots,p_M^1,x^1) \approx \Pi^1(p^1,x^1) + [\partial\Pi^1(p^1,x^1)/\partial p_1][p_1^0-p_1^1]$$
$$= p^1\cdot y^1 + y_1^1[p_1^0-p_1^1] \qquad \text{using (2) and (3).}$$

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[15] Grifell-Tatjé and Lovell (2015) devote many pages to alternative approaches to this distribution problem. They note the early contributions of Davis (1947) to this problem and his insights into many other aspects of productivity measurement. Grifell-Tatjé and Lovell do an excellent job on covering the history of productivity measurement and its connection with accounting theory. See also Lawrence, Diewert and Fox (2006) for a related exact index number application of this type of distributive analysis.

[16] As mentioned in the introduction, we are looking for a difference counterpart to the multiplicative decomposition of aggregate output price change into individual output price and input quantity change components that Diewert and Morrison (1986; 666-667) and Kohli (1990) obtained in their traditional index number approach to the decomposition of value added growth.

Thus we have the following first order approximation to the first term on the right hand side of (23):

$$(25)\ \Pi^1(p^1,x^1) - \Pi^1(p_1^{\,0},p_2^{\,1},..., p_M^{\,1},x^1) \approx p^1\cdot y^1 - \{p^1\cdot y^1 + y_1^{\,1}[p_1^{\,0}-p_1^{\,1}]\} \quad \text{using (2) and (24)}$$
$$= y_1^{\,1}[p_1^{\,1}-p_1^{\,0}].$$

In a similar manner to the derivation of (24), we can derive the following first order approximation to $\Pi^1(p_1^{\,0},p_2^{\,0},p_3^{\,1},..., p_M^{\,1},x^1)$:

$$(26)\ \Pi^1(p_1^{\,0},p_2^{\,0},p_3^{\,1},..., p_M^{\,1},x^1)$$
$$\approx \Pi^1(p^1,x^1) + [\partial\Pi^1(p^1,x^1)/\partial p_1][p_1^{\,0}-p_1^{\,1}] + [\partial\Pi^1(p^1,x^1)/\partial p_2][p_2^{\,0}-p_2^{\,1}]$$
$$= p^1\cdot y^1 + y_1^{\,1}[p_1^{\,0}-p_1^{\,1}] + y_2^{\,1}[p_2^{\,0}-p_2^{\,1}] \quad \text{using (2) and (3).}$$

Thus we have the following first order approximation to the second term on the right hand side of (23):

$$(27)\ \Pi^1(p_1^{\,0},p_2^{\,1},..., p_M^{\,1},x^1) - \Pi^1(p_1^{\,0},p_2^{\,0},p_3^{\,1},..., p_M^{\,1},x^1)$$
$$\approx p^1\cdot y^1 + y_1^{\,1}[p_1^{\,0}-p_1^{\,1}] - \{p^1\cdot y^1 + y_1^{\,1}[p_1^{\,0}-p_1^{\,1}] + y_2^{\,1}[p_2^{\,0}-p_2^{\,1}]\} \quad \text{using (24) and (26)}$$
$$= y_2^{\,1}[p_2^{\,1}-p_2^{\,0}].$$

In a similar fashion, it can be shown that the mth term on the right hand side of (23) has the first order approximation $y_m^{\,1}[p_m^{\,1}-p_m^{\,0}]$ for m = 1,2,...,M. The sum of these first order approximations is:

$$(28)\ \Sigma_{m=1}^{M}\ y_m^{\,1}[p_m^{\,1}-p_m^{\,0}] = y^1\cdot(p^1 - p^0) \approx \Pi^1(p^1,x^1) - \Pi^1(p^0,x^1) \quad \text{using (9).}$$

Thus we have decomposed the Paasche type measure of aggregate price change, $\Pi^1(p^1,x^1) - \Pi^1(p^0,x^1)$, into the sum of the M individual price change measures on the right hand side of (23) and the mth individual price change measure is approximately equal to $y_m^{\,1}[p_m^{\,1}-p_m^{\,0}]$ for m = 1,2,...,M.

Recall that the decomposition of value added growth defined by (8) had the overall output price change term $\Pi^0(p^1,x^0) - \Pi^0(p^0,x^0)$ on the right hand side of the equation. We want to decompose this Laspeyres type overall measure of price change into separate output price change contributions. We use the following decomposition:

$$(29)\ \Pi^0(p^1,x^0) - \Pi^0(p^0,x^0) = \Pi^0(p^1,x^0) - \Pi^0(p_1^{\,0},p_2^{\,1},..., p_M^{\,1},x^0)$$
$$+ \Pi^0(p_1^{\,0},p_2^{\,1},..., p_M^{\,1},x^0) - \Pi^0(p_1^{\,0},p_2^{\,0},p_3^{\,1},..., p_M^{\,1},x^0)$$
$$+ \ldots$$
$$+ \Pi^0(p_1^{\,0},...,p_{M-1}^{\,0},p_M^{\,1},x^0) - \Pi^0(p^0,x^0).$$

The right hand side of (29) consists of M differences in $\Pi^0(p,x^0)$ where each difference changes only one component of the p vector. We will approximate these terms by taking first order Taylor series approximations to the $\Pi^0(p,x^0)$ around the point $p = p^0$. Thus the first order approximation to $\Pi^0(p_1^{\,0},...,p_{M-1}^{\,0},p_M^{\,1},x^0)$ is the following one:

$$(30)\ \Pi^0(p_1^0,...,p_{M-1}^0,p_M^1,x^0) \approx \Pi^0(p^0,x^0) + [\partial\Pi^0(p^0,x^0)/\partial p_M][p_M^1 - p_M^0]$$
$$= p^0\cdot y^0 + y_M^0[p_M^1 - p_M^0] \qquad \text{using (2) and (3).}$$

Thus we have the following first order approximation to the last term on the right hand side of (29):

$$(31)\ \Pi^0(p_1^0,...,p_{M-1}^0,p_M^1,x^0) - \Pi^0(p^0,x^0) \approx p^0\cdot y^0 + y_M^0[p_M^1 - p_M^0] - p^0\cdot y^0 \quad \text{using (2) and (30)}$$
$$= y_M^0[p_M^1 - p_M^0].$$

In a similar fashion, it can be shown that the mth term on the right hand side of (29) has the first order approximation $y_m^0[p_m^1-p_m^0]$ for m = 1,2,...,M. The sum of these first order approximations is:

$$(32)\ \Sigma_{m=1}^M\ y_m^0[p_m^1 - p_m^0] = y^0\cdot(p^1 - p^0) \approx \Pi^0(p^1,x^0) - \Pi^0(p^0,x^0) \qquad \text{using (13).}$$

Thus we have decomposed the Laspeyres type measure of aggregate price change, $\Pi^0(p^1,x^0) - \Pi^0(p^0,x^0)$, into the sum of the M individual price change measures on the right hand side of (29) and the mth individual price change measure is approximately equal to $y_m^0[p_m^1-p_m^0]$ for m = 1,2,...,M.

Recall that the overall Bennet indicator of value added price change was defined as $(1/2)(y^0+y^1)\cdot(p^1-p^0)$ which is the arithmetic average of the Paasche and Laspeyres measures of price change, $\Sigma_{m=1}^M\ y_m^1[p_m^1-p_m^0]$ and $\Sigma_{m=1}^M\ y_m^0[p_m^1-p_m^0]$ respectively. Thus the mth term in the Bennet indicator of value added price change, $(1/2)y_m^1[p_m^1-p_m^0] + (1/2)y_m^0[p_m^1-p_m^0]$, can be interpreted as an approximation to the theoretical measure of change in the price of the mth output that is defined by the arithmetic average of the mth terms on the right hand sides of (23) and (29).

The decomposition of value added growth defined by (7) had the overall input quantity change term $[\Pi^0(p^0,x^1)-\Pi^0(p^0,x^0)]$ on the right hand side of the equation. We want to decompose this Laspeyres type overall measure of input quantity change into separate input quantity change contributions. We use the following decomposition:

$$(33)\ \Pi^0(p^0,x^1) - \Pi^0(p^0,x^0) = \Pi^0(p^0,x^1) - \Pi^0(p^0,x_1^0,x_2^1,...,x_N^1)$$
$$+ \Pi^0(p^0,x_1^0,x_2^1,...,x_N^1) - \Pi^0(p^0,x_1^0,x_2^0,x_3^1...,x_N^1)$$
$$+ \ldots$$
$$+ \Pi^0(p^0,x_1^0,...,x_{N-2}^0, x_{N-1}^1,x_N^1) - \Pi^0(p^0,x_1^0,...,x_{N-1}^0,x_N^1)$$
$$+ \Pi^0(p^0,x_1^0,...x_{N-1}^0,x_N^1) - \Pi^0(p^0,x^0)$$

The right hand side of (33) consists of N differences in $\Pi^0(p^0,x)$ where each difference changes only one component of the x vector. We will approximate these terms by taking first order Taylor series approximations to the $\Pi^0(p^0,x)$ around the point $x = x^0$. Thus the first order approximation to $\Pi^0(p^0,x_1^0,...x_{N-1}^0,x_N^1)$ is the following one:

$$(34)\ \Pi^0(p^0,x_1^0,...x_{N-1}^0,x_N^1) \approx \Pi^0(p^0,x^0) + [\partial\Pi^0(p^0,x^0)/\partial x_N][x_N^1-x_N^0]$$
$$= p^0\cdot y^0 + w_N^0[x_N^1-x_N^0] \qquad \text{using (2) and (4).}$$

Thus we have the following first order approximation to the last term on the right hand side of (33):

$$(35)\ \Pi^0(p^0,x_1^0,...x_{N-1}^0,x_N^1) - \Pi^0(p^0,x^0) \approx p^0\cdot y^0 + w_N^0[x_N^1-x_N^0] - p^0\cdot y^0 \quad \text{using (2) and (34)}$$
$$= w_N^0[x_N^1-x_N^0].$$

In a similar fashion, it can be shown that the nth term on the right hand side of (33) has the first order approximation $w_n^0[x_n^1-x_n^0]$ for n = 1,2,...,N. Thus the sum of these first order approximations is:

$$(36)\ \Sigma_{n=1}^N\ w_n^0[x_n^1-x_n^0] = w^0\cdot(x^1 - x^0) \approx \Pi^0(p^0,x^1) - \Pi^0(p^0,x^0) \qquad \text{using (11).}$$

Thus we have decomposed the Laspeyres type measure of aggregate input quantity change, $\Pi^0(p^0,x^1) - \Pi^0(p^0,x^0)$, into the sum of the N individual input quantity change measures on the right hand side of (33) and the nth individual quantity change measure is approximately equal to $w_n^0[x_n^1-x_n^0]$ for n = 1,2,...,N.

The decomposition of value added growth defined by (8) had the overall input quantity change term $[\Pi^1(p^1,x^1)-\Pi^1(p^1,x^0)]$ on the right hand side of the equation. We want to decompose this Paasche type overall measure of input quantity change into individual input quantity change contributions. We use the following decomposition:

$$(37)\ \Pi^1(p^1,x^1) - \Pi^1(p^1,x^0) = \Pi^1(p^1,x^1) - \Pi^1(p^1,x_1^0,x_2^1,...,x_N^1)$$
$$+ \Pi^1(p^1,x_1^0,x_2^1,...,x_N^1) - \Pi^1(p^1,x_1^0,x_2^0,x_3^1...,x_N^1)$$
$$+ ...$$
$$+ \Pi^1(p^1,x_1^0,...x_{N-1}^0,x_N^1) - \Pi^1(p^1,x^0).$$

The right hand side of (37) consists of N differences in $\Pi^1(p^1,x)$ where each difference changes only one component of the x vector. As usual, we approximate these terms by taking first order Taylor series approximations to the $\Pi^1(p^1,x)$ around the point $x = x^1$. Thus the observable first order approximation to the unobservable term $\Pi^1(p^1,x_1^0,x_2^1,...,x_N^1)$ is the following one:

$$(38)\ \Pi^1(p^1,x_1^0,x_2^1,...,x_N^1) \approx \Pi^1(p^1,x^1) + [\partial\Pi^1(p^1,x^1)/\partial x_1][x_1^0-x_1^1]$$
$$= p^1\cdot y^1 + w_1^1[x_1^0-x_1^1] \qquad \text{using (2) and (4).}$$

Thus we have the following observable first order approximation to the unobservable first term on the right hand side of (37):

$$(39)\ \Pi^1(p^1,x^1) - \Pi^1(p^1,x_1^0,x_2^1,...,x_N^1) \approx p^1\cdot y^1 - \{p^1\cdot y^1 + w_1^1[x_1^0-x_1^1]\} \qquad \text{using (2) and (38)}$$
$$= w_1^1[x_1^1-x_1^0].$$

In a similar fashion, it can be shown that the nth term on the right hand side of (37) has the first order approximation $w_n^1[x_n^1 - x_n^0]$ for n = 1,2,...,N. Thus the sum of these first order approximations is:

$$(40)\ \Sigma_{n=1}^{N}\ w_n^1[x_n^1 - x_n^0] = w^1 \cdot (x^1 - x^0) \approx \Pi^1(p^1,x^1) - \Pi^1(p^1,x^0) \qquad \text{using (15).}$$

Thus we have decomposed the Paasche type measure of aggregate input quantity change, $\Pi^1(p^1,x^1) - \Pi^1(p^1,x^0)$, into the sum of the N individual input quantity change measures defined on the right hand side of (37) and the nth individual quantity change measure is approximately equal to $w_n^1[x_n^1 - x_n^0]$ for n = 1,2,...,N.

Recall that the overall Bennet indicator of input quantity change was defined as $(1/2)(w^0 + w^1) \cdot (x^1 - x^0)$ which is the arithmetic average of the Laspeyres and Paasche measures of input quantity change, $\Sigma_{n=1}^{N}\ w_n^0[x_n^1 - x_n^0]$ and $\Sigma_{n=1}^{N}\ w_n^1[x_n^1 - x_n^0]$ respectively. Thus the nth term in the Bennet indicator of aggregate input quantity change, $(1/2)w_n^0[x_n^1 - x_n^0] + (1/2)w_n^1[x_n^1 - x_n^0]$, can be interpreted as an approximation to the theoretical measure of input n quantity change that is defined by the arithmetic average of the nth terms on the right hand sides of (33) and (37).

Hence, using the results of this section and any one of our four expressions of technical change, $B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1)$, from (18), (19) and (22), (17) can be re-written as follows:

$$\begin{aligned}(41)\ p^1 \cdot y^1 - p^0 \cdot y^0 &= (1/2)(y^0 + y^1) \cdot (p^1 - p^0) + B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1) \\ &\quad + (1/2)(w^0 + w^1) \cdot (x^1 - x^0) \\ &= \Sigma_{m=1}^{M}\ \{(1/2)y_m^1[p_m^1 - p_m^0] + (1/2)y_m^0[p_m^1 - p_m^0]\} \\ &\quad + B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1) \\ &\quad + \Sigma_{n=1}^{N}\ \{(1/2)w_n^0[x_n^1 - x_n^0] + (1/2)w_n^1[x_n^1 - x_n^0]\}.\end{aligned}$$

Thus, we have the contribution of each output to overall output price change, and the contribution of each input to overall input quantity change, which also represent their respective contributions to value change.

4. The Problem of Adjusting the Measures for General Inflation

"Today's dollar is, then, a totally different unit from the dollar of 1897. As the general price level fluctuates, the dollar is bound to become a unit of different magnitude. To mix these units is like mixing inches and centimeters or measuring a field with a rubber tape-line."

Livingston Middleditch (1918: 114–115).

Diewert (2005; 339) noted the above quotation by Middleditch in his discussion of the problem of adjusting for general inflation in making revenue, cost and profit comparisons in difference form over two periods in time. Diewert noted that if there is a great change in the general purchasing power of money between the two periods being compared, then the Bennet indicators of quantity change may be "excessively" heavily weighted by the prices of the period with the highest general price level. His solution to this weighting problem was very simple: in each period, divide the period t nominal output and input prices, p_m^t and w_n^t by a suitable price index, say ρ^t. Thus define the period t *real output and input price vectors*, p^{t*} and w^{t*} as follows:

$$(42)\ p^{t*} \equiv p^t/\rho^t\ ;\ w^{t*} \equiv w^t/\rho^t\ ; \qquad t = 0,1,...,T.$$

The period t value added function $\Pi^t(p,x)$ is linearly homogeneous in the components of p as are the derivatives $\partial\Pi^t(p,x)/\partial x_n$ for n = 1,...,T. Using these homogeneity properties, we can establish the following counterparts to the Hotelling and Samuelson Lemma results (2) and (3):

$$(43)\ \nabla_p\Pi^t(p^{t*},x^t) \equiv \nabla_p\Pi^t(p^t/\rho^t,x^t) = \nabla_p\Pi^t(p^t,x^t) = y^t\ ; \qquad t = 0,1,...,T;$$
$$(44)\ \nabla_x\Pi^t(p^{t*},x^t) \equiv \nabla_x\Pi^t(p^t/\rho^t,x^t) = (1/\rho^t)\nabla_x\Pi^t(p^t,x^t) = (1/\rho^t)w^t = w^{t*}\ ; \qquad t = 0,1,...,T.$$

Thus all of the nonparametric results established in Sections 2 and 3 above go through unchanged if we replace p^t by p^{t*} and w^t by w^{t*}. The significance of this result is substantial: if we deflate nominal prices into real prices in each time period, it is almost certain that real price change from period to period will be less than the corresponding nominal price change. *Thus the first order approximations used in the previous sections will generally be subject to smaller errors and hence our decompositions using real prices are going to be more accurate*. This is particularly true if between period inflation is high. This is a powerful argument for using real prices.

There remains the problem of choosing the deflator, ρ^t. In order to determine an appropriate deflator, we need to ask what is the purpose of the analysis or what is the application of the theory? In most applications, the task at hand is the measurement of the productivity growth of the production unit under consideration. If the production unit is a firm, investors will be interested in revenues and costs deflated by a suitable consumer price index. Factors of production will be interested in the growth of their real compensation; i.e., how many bundles of consumption can their present period compensation purchase relative to the previous period. Again, deflation by a consumer price index seems appropriate. Some policy makers may argue for a broader deflator such as a deflator for domestic output or absorption. In summary, the choice of the deflator will depend on the purpose of the exercise but in most cases, deflation by a consumer price index (or a consumption deflator) will probably be appropriate.

The analysis presented up to this point suffers from (at least) two problems:

- We have assumed constant returns to scale and
- We have assumed (competitive) profit maximizing behavior and hence there is no possibility of technical inefficiency.

In the following section, we develop an alternative methodology that allows for the possibility of technical inefficiency.[17]

5. The Difference Approach to Productivity Measurement Using the Nonparametric Cost Constrained Value Added Function

Diewert and Fox (2018b) worked out an approach to the measurement of productivity in a constant returns to scale context that was based on traditional multiplicative index number theory.[18] The theoretical indexes used in that paper could be calculated using the concept of a cost constrained value added function that made use of a particular nonparametric approximation to the true technology of a production unit. The nonparametric approximation to the true technology is the set of all nonnegative linear combinations of past production vectors. This section reworks their multiplicative index number decompositions into difference form. The details follow.

Define the production unit's *period t cost constrained value added function*, $R^t(p,w,x)$ as follows:[19]

$$(45)\ R^t(p,w,x) \equiv \max_{y,z} \{p\cdot y : (y,z)\in S^t;\ w\cdot z \leq w\cdot x\}.$$

If (y^*,z^*) solves the constrained maximization problem defined by (45), then the value added $p\cdot y$ of the production unit is maximized subject to the constraints that (y,z) is a feasible production vector and primary input expenditure $w\cdot z$ is equal to or less than "observed" primary input expenditure $w\cdot x$. Thus if the sector faces the prices $p^t >> 0_M$ and $w^t >> 0_N$ during period t and (y^t, x^t) is the sector's observed production vector, then production will be *value added efficient* if the observed value added, $p^t\cdot y^t$, is equal to the optimal value added, $R^t(p^t,w^t,x^t)$. However, production may not be efficient and so the following inequality will hold:

$$(46)\ p^t\cdot y^t \leq R^t(p^t,w^t,x^t)\ ; \qquad t = 0,1,\ldots,T.$$

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[17] We are not able to relax the assumption of constant returns to scale because the nonparametric cost constrained value added function that we use in our analysis in the following section is not always well defined unless we assume constant returns to scale in production.

[18] This paper drew heavily on the earlier papers by Balk (2003) and Diewert (2014).

[19] The cost constrained value added function is analogous to Diewert's (1983; 1086) *balance of trade restricted value added function* and Diewert and Morrison's (1986; 669) *domestic sales function*. However, the basic idea can be traced back to Shephard's (1974) *maximal return function*, Fisher and Shell's (1998; 48) *cost restricted sales function* and Balk's (2003; 34) *indirect revenue function*. See also Färe, Grosskopf and Lovell (1992; 286) and Färe and Primont (1994; 203) on Shephard's formulation. Shephard, Fisher and Shell and Balk defined their functions as $IR^t(p,w,c) \equiv \max_{y,z} \{p\cdot y : w\cdot z \leq c\ ;\ (y,z)\in S^t\}$ where $c > 0$ is a scalar cost constraint. It can be seen that our cost constrained value added function replaces c in the above definition by $w\cdot x$, a difference which will be important in forming our input indicators and hence our value added decompositions. Another difference is that our y vector is a net output vector; i.e., some components of y can be negative. With the exceptions of Diewert and Morrison (1986) and Diewert (1983), the other authors required that y be nonnegative. This makes a difference to our analysis. Also, our regularity conditions are weaker than the ones that are usually used.

Adapting the ratio definition of Balk (1998; 143) to the difference context, we define the *value added* or *net revenue efficiency* of the production unit during period t, e^t, as follows:

$$(47)\ e^t \equiv p^t \cdot y^t - R^t(p^t,w^t,x^t) \leq 0\ ; \qquad t = 0,1,...,T$$

where the inequality in (47) follows from (46). Thus if $e^t = 0$, then production is allocatively efficient in period t and if $e^t < 0$, then production during period t is allocatively inefficient. Note that the above definition of value added efficiency is a net revenue difference counterpart to Farrell's (1957; 255) cost based measure of *overall efficiency* in the DEA context, which combined his measures of technical and (cost) allocative efficiency. DEA or *Data Envelopment Analysis* is the term used by Charnes and Cooper (1985) and their co-workers to denote an area of analysis which is called the nonparametric approach to production theory or the measurement of the efficiency of production by economists.[20]

We assume that the production unit's period t production possibilities set S^t is the conical free disposal hull of the period t actual production vector and past production vectors that are in our sample of time series observations for the unit.[21] Using this assumption about S^t, for strictly positive price vectors p and w and nonnegative input quantity vector x, we define the *period t cost constrained value added function* $R^t(p,w,x)$ for the production unit as follows:

$$\begin{aligned}(48)\ R^t(p,w,x) &\equiv \max_{y,z} \{p\cdot y : w\cdot z \leq w\cdot x\ ;\ (y,z)\in S^t\} \\ &\geq \max_{\lambda} \{p\cdot\lambda y^s : w\cdot\lambda x^s \leq w\cdot x;\ \lambda \geq 0\} \qquad \text{since } (\lambda y^s,\lambda x^s)\in S^t \text{ for all } \lambda \geq 0 \\ &= \max_{\lambda} \{\lambda p\cdot y^s : \lambda w\cdot x^s \leq w\cdot x;\ \lambda \geq 0\} \\ &= (w\cdot x/w\cdot x^s)p\cdot y^s.\end{aligned}$$

The inequality in (48) will hold for all s = 1,2,...,t. Thus we have:

$$(49)\ R^t(p,w,x) \geq \max_s \{p\cdot y^s\ w\cdot x/w\cdot x^s : s = 1,2,...,t\}.$$

The rays $(\lambda y^s,\lambda x^s)\in S^t$ for $\lambda \geq 0$ generate the efficient points in the set S^t so the strict inequality in (49) cannot hold and so we have:

$$\begin{aligned}(50)\ R^t(p,w,x) &\equiv \max_{y,z} \{p\cdot y : w\cdot z \leq w\cdot x\ ;\ (y,z)\in S^t\} \\ &= \max_s \{p\cdot y^s\ w\cdot x/w\cdot x^s : s = 1,2,...,t\}\end{aligned}$$

[20] The early contributors to this literature were Farrell (1957), Afriat (1972), Hanoch and Rothschild (1972), Färe and Lovell (1978), Diewert and Parkan (1983), Varian (1984) and Färe, Grosskopf and Lovell (1985).

[21] Diewert (1980; 264) suggested that the convex, conical, free disposal hull of past and current production vectors be used as an approximation to the period t technology set S^t when measuring TFP growth. Tulkens (1993; 201-206) and Diewert and Fox (2014) (2017) dropped the convexity and constant returns to scale assumptions and used free disposal hulls of past and current production vectors to represent the period t technology sets. In this paper, we also drop the convexity assumption but maintain the free disposal and constant returns to scale assumptions. We also follow Diewert and Parkan (1983; 153-157), Balk (2003; 37) and Diewert and Mendoza (2007) in introducing price data into the computations.

$$= w\cdot x \max_s\{p\cdot y^s/w\cdot x^s : s = 1,2,...,t\}$$
$$= \max_{\lambda_1,...,\lambda_t} \{p\cdot(\Sigma_{s=1}^t y^s\lambda_s) ; w\cdot(\Sigma_{s=1}^t x^s\lambda_s) \leq w\cdot x ; \lambda_1 \geq 0 ,..., \lambda_t \geq 0\}$$

where the last line in (50) follows from the fact that the solution to the linear programming problem is an extreme point and thus its solution is equal to the second line in (50). Thus all three equalities in (50) can serve to define $R^t(p,w,x)$. We assume that all inner products of the form $p\cdot y^s$ and $w\cdot x^s$ are positive and this assumption rules out the possibility of a $\lambda_s = 0$ solution to the third line in (50). The last expression in (50) can be used to show that when we assume constant returns to scale for our nonparametric representation for S^t, the resulting $R^t(p,w,x)$ is linear and nondecreasing in x, is convex and linearly homogeneous in p and is homogeneous of degree 0 in w. The bottom line is that the third equality in (50) can be used to evaluate the function $R^t(p,w,x)$ as p, w and x take on the observable values in the definitions which follow.

Our task in this section is to decompose the growth in observed nominal value added over the two periods, $p^t\cdot y^t - p^{t-1}\cdot y^{t-1}$, into explanatory growth factors.

One of the explanatory factors will be the *growth in the value added efficiency* of the sector or production unit. Above, we defined the period t value added efficiency as $e^t \equiv p^t\cdot y^t - R^t(p^t,w^t,x^t)$. Define the corresponding period t−1 efficiency as $e^{t-1} \equiv p^{t-1}\cdot y^{t-1} - R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$. Given the above definitions of value added efficiency in periods t−1 and t, we can define an index of the *change in value added efficiency* ε^t for the sector over the two periods as follows:

(51) $\varepsilon^t \equiv e^t - e^{t-1} = p^t\cdot y^t - p^{t-1}\cdot y^{t-1} - [R^t(p^t,w^t,x^t) - R^{t-1}(p^{t-1},w^{t-1},x^{t-1})]$; $t = 1,2,...,T.$

The above equations can be rewritten as follows:

(52) $p^t\cdot y^t - p^{t-1}\cdot y^{t-1} = \varepsilon^t + R^t(p^t,w^t,x^t) - R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$; $t = 1,2,...,T.$

Notice that the cost constrained value added function for the production unit in period t, $R^t(p,w,x)$, depends on four sets of variables:

- The time period t and this index t serves to indicate that the period t technology set S^t is used to define the period t value added function;
- The vector of net output prices p that the production unit faces;
- The vector of primary input prices w that the production unit faces and
- The vector of primary inputs x which is available for use by the production unit during period t.

At this point, we will follow the methodology that is used in the economic approach to index number theory that originated with Konüs (1939) and Allen (1949) and we will use the value added function to define various *families of indexes* that vary only *one* of the four sets of variables, t, p, w and x, between the two periods under consideration and hold constant the other sets of variables.

Our first family of factors that explain sectoral value added growth is a family of *net output price indicators*, $\alpha(p^{t-1},p^t,w,x,t)$:

(53) $\alpha(p^{t-1},p^t,w,x,s) \equiv R^s(p^t,w,x) - R^s(p^{t-1},w,x)$.

Following the example of Konüs (1939) in his analysis of the true cost of living index, it is natural to single out two special cases of the family of net output price indicators defined by (53): one choice where we use the period t−1 technology and set the reference input prices and quantities equal to the period t−1 input prices and quantities w^{t-1} and x^{t-1} (which gives rise to a *Laspeyres type net output price indicator*) and another choice where we use the period t technology and set the reference input prices and quantities equal to the period t prices and quantities w^t and x^t (which gives rise to a *Paasche type net output price indicator*). We define these special cases $\alpha_L{}^t$ and $\alpha_P{}^t$ for t = 1,...,T as follows:

(54) $\alpha_L{}^t \equiv \alpha(p^{t-1},p^t,w^{t-1},x^{t-1},t-1) \equiv R^{t-1}(p^t,w^{t-1},x^{t-1}) - R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$;
(55) $\alpha_P{}^t \equiv \alpha(p^{t-1},p^t,w^t,x^t,t) \equiv R^t(p^t,w^t,x^t) - R^t(p^{t-1},w^t,x^t)$.

Our second family of factors that explain value added growth is a family of *input quantity indicators*, $\beta(x^{t-1},x^t,p,w,s)$:

(56) $\beta(x^{t-1},x^t,p,w,s) \equiv R^s(p,w,x^t) - R^s(p,w,x^{t-1})$

It is natural to single out two special cases of the family of input quantity indexes defined by (56): one choice where we use the period t−1 technology, input prices and output prices as the reference p,w and s which gives rise to the *Laspeyres input quantity indicator* $\beta_L{}^t$ and another choice where we set the reference p, w equal to p^t and w^t and set s equal t which gives rise to the *Paasche input quantity indicator* $\beta_P{}^t$. Thus define these special cases $\beta_L{}^t$ and $\beta_P{}^t$ for t = 1,...,T as follows:

(57) $\beta_L{}^t \equiv R^{t-1}(p^{t-1},w^{t-1},x^t) - R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$;
(58) $\beta_P{}^t \equiv R^t(p^t,w^t,x^t) - R^t(p^t,w^t,x^{t-1})$.

Our next family of indexes will measure the effects on cost constrained value added of a change in input prices going from period t−1 to t. We consider a family of measures of the relative change in cost constrained value added of the form $R^s(p,w^t,x) - R^s(p,w^{t-1},x)$. Since $R^s(p,w,x)$ is homogeneous of degree 0 in the components of w, it can be seen that we cannot interpret $R^s(p,w^t,x)/R^s(p,w^{t-1},x)$ as an input price index and hence $R^s(p,w^t,x) - R^s(p,w^{t-1},x)$ cannot be interpreted as an input price indicator. It is best to interpret $R^s(p,w^t,x) - R^s(p,w^{t-1},x)$ as measuring the effects on cost constrained value added of a change in the relative proportions of inputs and outputs used in production or in the *mix* of inputs and outputs used in production that is induced by a change in relative input prices when there is more than one primary input. Thus define the family of *input mix indicators* $\gamma(w^{t-1},w^t,p,x,s)$ as follows:[22]

(59) $\gamma(w^{t-1},w^t,p,x,s) \equiv R^s(p,w^t,x) - R^s(p,w^{t-1},x)$.

We will consider two special cases of the above family of input mix indicators, neither of which is a "pure" Laspeyres or Paasche type indicator:

(60) $\gamma_{LP}{}^t \equiv \gamma(w^{t-1},w^t,p^{t-1},x^t,t) \equiv R^t(p^{t-1},w^t,x^t) - R^t(p^{t-1},w^{t-1},x^t)$; $t = 1,...,T$;
(61) $\gamma_{PL}{}^t \equiv \gamma(w^{t-1},w^t,p^t,x^{t-1},t-1) \equiv R^{t-1}(p^t,w^t,x^{t-1}) - R^{t-1}(p^t,w^{t-1},x^{t-1})$; $t = 1,...,T$.

The reason for these rather odd looking choices for reference vectors will become apparent below because they lead to exact decompositions of the difference in observed value added between two successive periods.

Finally, we use the cost constrained value added function in order to define a family of *technical progress indicators* going from period t−1 to t, $\tau(t,p,w,x)$, for reference vectors of output and input prices, p and w, and a reference vector of input quantities x as follows:[23]

(62) $\tau(t,p,w,x) \equiv R^t(p,w,x) - R^{t-1}(p,w,x)$.

If there is positive technical progress going from period t−1 to t, then $R^t(t,p,w,x)$ will generally be greater than $R^{t-1}(p,w,x)$ and hence $\tau(t,p,w,x)$ will be greater than zero. If S^{t-1} is a subset of S^t (so that technologies are not forgotten), then $\tau(t,p,w,x) \geq 0$.

Again, we will consider two special cases of the above family of technical progress indexes, a "mixed" Laspeyres case and a "mixed" Paasche case. The Laspeyres case $\tau_L{}^t$ will use the period t input vector x^t as the reference input vector and the period t−1 reference output and input price vectors p^{t-1} and w^{t-1} while the Paasche case $\tau_P{}^t$ will use the use the period t−1 input vector x^{t-1} as the reference input and the period t reference output and input price vectors p^t and w^t:

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[22] It would be more accurate to say that $\gamma(w^{t-1},w^t,p,x,s)$ represents the hypothetical change in cost constrained value added for the period s reference technology due to the effects of a change in the input price vector from w^{t-1} to w^t when facing the reference net output prices p and the reference vector of inputs x. Thus we shorten this description to say that γ is an "input mix indicator". If there is only one primary input, then since $R^s(p,w,x)$ is homogeneous of degree 0 in w, $R^s(p,w,x)$ does not vary as the scalar w varies and hence $\gamma(w^{t-1},w^t,p,x,s) \equiv 0$; i.e., if there is only one primary input, then the input mix index is identically equal to 0. For alternative mix definitions in the index number context, see Balk (2001) and Diewert (2014; 62).

[23] The counterpart to this family of technical progress indicators was defined in the index number context by Diewert and Morrison (1986; 662) using the value added function $\Pi^t(p,x)$. A special case of this ratio family was defined earlier by Diewert (1983; 1063). Balk (1998; 99) also used this definition and Balk (1998; 58), following the example of Salter (1960), also used the joint cost function to define a similar family of technical progress indexes.

$$(63)\ \tau_L{}^t \equiv \tau(t,p^{t-1},w^{t-1},x^t) \equiv R^t(p^{t-1},w^{t-1},x^t) - R^{t-1}(p^{t-1},w^{t-1},x^t).$$
$$(64)\ \tau_P{}^t \equiv \tau(t,p^t,w^t,x^{t-1}) \quad \equiv R^t(p^t,w^t,x^{t-1}) - R^{t-1}(p^t,w^t,x^{t-1}).$$

We are now in a position to decompose the growth in nominal value added for the production unit going from period t−1 to t as the sum of five explanatory indicators of change:

- The change in cost constrained value added efficiency over the two periods; i.e., $\varepsilon^t \equiv e^t - e^{t-1}$ defined by (51) above;
- Changes in net output prices; i.e., an indicator of the form $\alpha(p^{t-1},p^t,w,x,s)$ defined above by (53);
- Changes in input quantities; i.e., an indicator of the form $\beta(x^{t-1},x^t,p,w,s)$ defined by (56);
- Changes in input prices; i.e., an input mix indicator of the form $\gamma(w^{t-1},w^t,p,x,s)$ defined by (59) and
- Changes due to technical progress; i.e., an indicator of the form $\tau(t,p,w,x)$ defined by (62).

Straightforward algebra using the above definitions shows that we have the following exact decompositions of the observed value added difference going from period t−1 to t into explanatory indicators of the above type for t = 1,...,T:[24]

$$(65)\ p^t\cdot y^t - p^{t-1}\cdot y^{t-1} = \varepsilon^t + \alpha_P{}^t + \beta_L{}^t + \gamma_{LP}{}^t + \tau_L{}^t\ ;$$
$$(66)\ p^t\cdot y^t - p^{t-1}\cdot y^{t-1} = \varepsilon^t + \alpha_L{}^t + \beta_P{}^t + \gamma_{PL}{}^t + \tau_P{}^t\ .$$

Define the period t arithmetic averages of the above α, β, γ and τ indicators as follows for t = 1,...,T:

$$(67)\ \alpha^t \equiv (1/2)(\alpha_L{}^t+\alpha_P{}^t);\ \beta^t \equiv (1/2)(\beta_L{}^t+\beta_P{}^t);\ \gamma^t \equiv (1/2)(\gamma_{LP}{}^t+\gamma_{PL}{}^t);\ \tau^t \equiv (1/2)(\tau_L{}^t+\tau_P{}^t).$$

Each of the exact decompositions defined by (65) and (66) gives a somewhat different picture of the growth process. If we take the arithmetic average of these decompositions, we will obtain a decomposition that will give the same results whether we measure time going forward or backwards. Hence our preferred growth decomposition is the following one which averages the two decompositions:

$$(68)\ p^t\cdot y^t - p^{t-1}\cdot y^{t-1} = \varepsilon^t + \alpha^t + \beta^t + \gamma^t + \tau^t; \qquad t = 1,\ldots,T.$$

Following Jorgenson and Griliches (1967), a Total Factor Productivity (TFP) Growth index can be defined as an output quantity index divided by an input quantity index.[25] Translating this concept into the difference context, we define a TFP indicator as an

[24] These decompositions are the difference analogues to the ratio decompositions obtained by Diewert and Fox (2018b).
[25] This definition of TFP growth can be traced back to Copeland (1937; 31); see Grifell-Tatjé and Lovell (2015; 69) for additional references to the early literature on definitions of TFP growth.

output quantity indicator less an input quantity indicator. An implicit output quantity indicator is value added growth less an output price indicator. Thus we define our *TFP indicator for period t* as follows:[26]

(69) $TFP^t \equiv p^t \cdot y^t - p^{t-1} \cdot y^{t-1} - \alpha^t - \beta^t$; $\qquad t = 1,...,T$
$= \varepsilon^t + \gamma^t + \tau^t \qquad$ using (67).

Thus the indicator of period t Total Factor Productivity growth, TFP^t, is equal to the sum of period t value added efficiency change ε^t, the period t input mix indicator γ^t (which typically will be close to 0)[27] and the period t indicator of technical progress τ^t. All of the terms in (69) can be measured under our assumptions on the technology sets S^t. The advantage of the decomposition of TFP growth defined by (69) compared to the decomposition (20) that was defined in section 2 above is that the technical change indicator τ^t that appears in (69) is always nonnegative whereas the Bennet indicator of technical progress $B_\tau(p^0,p^1,w^0,w^1,y^0,y^1,x^0,x^1)$ defined by (20) will usually become negative when there is a severe recession. Using the decomposition defined by (69) will avoid this problem: when there is a recession, the efficiency indicator ε^t will typically become negative, indicating that the production unit is no longer on its production frontier. Put another way, the approach outlined in section 2 assumes that the observed output and input vectors for period t, y^t and x^t, are always on the frontier of the period t production possibilities set S^t. This assumption is not plausible during recessions because firms cannot instantaneously dispose of their fixed inputs (land and structures) and they often employ more labour input than is efficient because it is not costless to fire and then rehire workers when the recession ends.

It is possible to decompose the overall output price indicator defined by the difference $R^s(p^t,w,x) - R^s(p^{t-1},w,x)$ into a sum of M commodity specific price indicators if we use the same type of decomposition of $\Pi^0(p^1,x^0) - \Pi^0(p^0,x^0)$ into individual price change components that was defined by (29) in section 3. Similarly, it is possible to decompose the overall input quantity indicator defined by the difference $R^s(p,w,x^t) - R^s(p,w,x^{t-1})$ into a sum of N commodity specific quantity indicators if we use the same type of decomposition of $\Pi^0(p^0,x^1) - \Pi^0(p^0,x^0)$ into individual quantity change components that was defined by (33) in section 3.

Finally, our discussion at the end of section 4 on the usefulness of replacing the nominal price vectors, p^t and w^t, by their deflated counterparts, $p^t/\rho^t \equiv p^{t*}$ and $w^t/\rho^t \equiv w^{t*}$, is still relevant in the present context (where ρ^t is a suitable period t deflator). Using the approach outlined in this section, we no longer have to worry about the accuracy of first order approximations since under our assumptions, we can compute all manner of hypothetical net revenues using the formula for $R^s(p,w,x)$ defined by the third equation in (49). However, the Middleditch quotation is still relevant; it does not make sense to

[26] The difference decomposition defined by (69) is the counterpart to the ratio type decomposition that was obtained by Diewert and Fox (2018b).
[27] In the empirical estimates made by Diewert and Fox (2018b), the mix index counterpart to our present indicator γ^t was always close to 1, implying that its difference counterpart will be close to 0.

compare nominal amounts of money across time periods when there is general inflation. Thus we recommend that the deflated prices, p^{t*} and w^{t*}, be used in place of the nominal prices, p^t and w^t, in the above definitions and decompositions in order to obtain more meaningful difference type comparisons.

In the following section, we outline our final approach to decomposing value added change into explanatory components.[28]

6. An Exact Indicator Approach to the Decomposition of Value Added Change

Suppose the period t value added function has the following *normalized quadratic* functional form:[29]

$$(70)\ \Pi^t(p,x) \equiv (1/2)p^TAp\,(\alpha\cdot p)^{-1}(\beta\cdot x) + (1/2)x^TBx\,(\alpha\cdot p)(\beta\cdot x)^{-1} + p^TCx + (a\cdot p)(\beta\cdot x)t + (\alpha\cdot p)(b\cdot x)t$$

where A is an M by M symmetric positive semidefinite matrix of parameters, B is a symmetric N by N matrix of parameters where B has one positive eigenvalue and N−1 nonpositive eigenvalues, C is an M by N matrix of parameters, a and $\alpha > 0_M$ are M dimensional vectors of parameters and b and $\beta > 0_N$ are N dimensional vectors of parameters. It can be shown that the Π defined by (70) is a flexible functional form (in the class of functional forms that are dual to technology sets that are subject to constant returns to scale) for a twice continuously differentiable value added function for any predetermined α and β vectors. Moreover, this functional form allows for commodity specific biased technical charge (the a and b parameter vectors accomplish this). We note that $\Pi^t(\lambda p,x) = \lambda\Pi^t(p,x)$ and $\Pi^t(p,\lambda x) = \lambda\Pi^t(p,x)$ for all scalars $\lambda > 0$; i.e., $\Pi^t(p,x)$ is linearly homogeneous in the components of p and x separately.

The term $(\alpha\cdot p)$ can be regarded as a fixed basket price index and the term $(\beta\cdot x)$ can be regarded as a constant weights input quantity index. We use these indexes to form the *normalized price and quantity vectors*, ρ and χ:[30]

$$(71)\ \rho^t \equiv p^t/\alpha\cdot p^t\ ;\ \chi^t \equiv x^t/\beta\cdot x^t\ ; \qquad t = 0,1.$$

Using the linear homogeneity properties of $\Pi^t(p^t,x^t)$ and definitions (71), it can be seen that $\Pi^t(\rho^t, \chi^t)$ is equal to the following expression:

$$(72)\ \Pi^t(\rho^t, \chi^t) = (1/2)\rho^t\cdot A\rho^t + (1/2)\,\chi^t\cdot B\chi^t + \rho^t\cdot C\chi^t + (a\cdot\rho^t)t + (b\cdot\chi^t)t\ ; \qquad t = 0,1.$$

[28] Balk, Färe and Grosskopf (2004) also used related techniques in an attempt to obtain an exact economic decomposition of a cost difference into Bennet type explanatory factors.
[29] See Diewert and Wales (1987) (1992) for applications of the normalized quadratic functional form to production theory.
[30] The new definition for ρ^t is different from the previous definition for ρ^t.

It can be seen that $\Pi(\rho, \chi^t, t)$ is a quadratic function in ρ, χ and t. Thus we have the following identities:[31]

$$(73)\ [\Pi^0(\rho^1, \chi^0) - \Pi^0(\rho^0, \chi^0)] + [\Pi^1(\rho^1, \chi^1) - \Pi^1(\rho^0, \chi^1)] = [\nabla_\rho \Pi^0(\rho^0, \chi^0) + \nabla_\rho \Pi^1(\rho^1, \chi^1)] \cdot [\rho^1 - \rho^0]\ ;$$

$$(74)\ [\Pi^0(\rho^0, \chi^1) - \Pi^0(\rho^0, \chi^0)] + [\Pi^1(\rho^1, \chi^1) - \Pi^1(\rho^1, \chi^0)] = [\nabla_\chi \Pi^0(\rho^0, \chi^0) + \nabla_\chi \Pi^1(\rho^1, \chi^1)] \cdot [\chi^1 - \chi^0]$$

We can evaluate the derivatives in (73) and (74) using observed data plus a knowledge of the parameter vectors α and β. Using Hotelling's Lemma, we have:

$$(75)\ y^t = \nabla_\rho \Pi^t(p^t, x^t)\ ; \qquad t = 0,1.$$

Using Samuelson's Lemma, we have:

$$(76)\ w^t = \nabla_x \Pi^t(p^t, x^t)\ ; \qquad t = 0,1.$$

Using (75) and (76), definitions (71) and the homogeneity properties of Π, we can establish the following results:

$$(77)\ \nabla_\rho \Pi^t(\rho^t, \chi^t) = \nabla_p \Pi^t(p^t, x^t/\beta \cdot x^t) = (\beta \cdot x^t)^{-1} \nabla_p \Pi^t(p^t, x^t) = y^t/\beta \cdot x^t\ ; \qquad t = 0,1;$$
$$(78)\ \nabla_\chi \Pi^t(\rho^t, \chi^t) = \nabla_x \Pi^t(p^t/\alpha \cdot p^t, x^t) = (\alpha \cdot p^t)^{-1} \nabla_x \Pi^t(p^t, x^t) = w^t/\alpha \cdot p^t\ ; \qquad t = 0,1.$$

(77) and (78) and the homogeneity properties of $\Pi^t(p,x)$ also imply the following relations:

$$(79)\ \Pi^t(\rho^t, \chi^t) = p^t \cdot y^t/(\alpha \cdot p^t \beta \cdot x^t) = \rho^t \cdot y^t/\beta \cdot x^t = w^t \cdot x^t/(\alpha \cdot p^t \beta \cdot x^t) = w^t \cdot \chi^t/\alpha \cdot p^t\ ; \qquad t = 0,1.$$

It is convenient to define the inflation adjusted input prices for period t, w^{t*}, as the unadjusted prices w^t divided by the exogenous price index for period t, $\alpha \cdot p^t$.[32] It is also convenient to define the normalized net output quantity vector for period t, y^{t*}, as the unadjusted net output vector y^t divided by the exogenous input index, $\beta \cdot x^t$. Thus we have:

$$(80)\ w^{t*} \equiv w^t/\alpha \cdot p^t\ ;\ y^{t*} \equiv y^t/\beta \cdot x^t\ ; \qquad t = 0,1.$$

Using definitions (80), equations (77)-(79) simplify to the following equations:

$$(81)\ \nabla_\rho \Pi^t(\rho^t, \chi^t) = y^{t*}\ ; \qquad t = 0,1;$$
$$(82)\ \nabla_\chi \Pi^t(\rho^t, \chi^t) = w^{t*}\ ; \qquad t = 0,1;$$
$$(83)\ \Pi^t(\rho^t, \chi^t) = \rho^t \cdot y^{t*} = w^{t*} \cdot \chi^t\ ; \qquad t = 0,1.$$

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[31] This identity is a generalization of Diewert's (1976; 118) *quadratic identity*. A logarithmic version of the above identity corresponds to the *translog identity* which was established in the Appendix to Caves, Christensen and Diewert (1982; 1412-1413).

[32] This definition for w^{t*} is also different from the definition used in the previous section.

We will call $\rho^t \cdot y^{t*} = w^{t*} \cdot \chi^t$ the *period t normalized value added* for the production unit. It is equal to unnormalized period t value added, $p^t \cdot y^t = w^t \cdot x^t$, divided by $\alpha \cdot p^t \beta \cdot x^t$.

Substituting (77) and (78) into (73) and (74) and using (80)-(83) leads to the following identities for the *Bennet indicators of normalized output price change and normalized input quantity change*:

$$(84)\ (1/2)[\Pi^0(\rho^1, \chi^0) - \Pi^0(\rho^0, \chi^0)] + (1/2)[\Pi^1(\rho^1, \chi^1) - \Pi^1(\rho^0, \chi^1)]$$
$$= (1/2)[(y^0/\beta \cdot x^0) + (y^1/\beta \cdot x^1)] \cdot [(p^1/\alpha \cdot p^1) - (p^0/\alpha \cdot p^0)]$$
$$= (1/2)[y^{0*} + y^{1*}] \cdot [\rho^1 - \rho^0]$$
$$\equiv B_\rho(\rho^0, \rho^1, y^{0*}, y^{1*})\ ;$$

$$(85)\ (1/2)[\Pi^0(\rho^0, \chi^1) - \Pi^0(\rho^0, \chi^0)] + (1/2)[\Pi^1(\rho^1, \chi^1) - \Pi^1(\rho^1, \chi^0)]$$
$$= (1/2)[(w^0/\alpha \cdot p^0) + (w^1/\alpha \cdot p^1)] \cdot [(x^1/\beta \cdot x^1) - (x^0/\beta \cdot x^0)]$$
$$= (1/2)[w^{0*} + w^{1*}] \cdot [\chi^1 - \chi^0]$$
$$\equiv B_\chi(\chi^0, \chi^1, w^{0*}, w^{1*}).$$

Recall the identities that were defined by equations (7) and (8). Similar decompositions can be applied to the normalized value added difference, $\Pi^1(\rho^1,\chi^1) - \Pi^0(\rho^0,\chi^0)$. Taking the arithmetic average of these two decompositions leads to the following decomposition:

$$(86)\ \Pi^1(\rho^1,\chi^1) - \Pi^0(\rho^0,\chi^0) = (1/2)[\Pi^1(\rho^1,\chi^1) - \Pi^1(\rho^0,\chi^1)] + (1/2)[\Pi^0(\rho^1,\chi^0) - \Pi^0(\rho^0,\chi^0)]$$
$$+ (1/2)[\Pi^1(\rho^0,\chi^1) - \Pi^0(\rho^0,\chi^1)] + (1/2)[\Pi^1(\rho^1,\chi^0) - \Pi^0(\rho^1,\chi^0)]$$
$$+ (1/2)[\Pi^0(\rho^0,\chi^1) - \Pi^0(\rho^0,\chi^0)] + (1/2)[\Pi^1(\rho^1,\chi^1) - \Pi^1(\rho^1,\chi^0)]$$
$$= (1/2)[y^{0*} + y^{1*}] \cdot [\rho^1 - \rho^0] + (1/2)[\Pi^1(\rho^0,\chi^1) - \Pi^0(\rho^0,\chi^1)] + (1/2)[\Pi^1(\rho^1,\chi^0) - \Pi^0(\rho^1,\chi^0)]$$
$$+ (1/2)[w^{0*} + w^{1*}] \cdot [\chi^1 - \chi^0] \qquad \text{using (83) and (84)}$$
$$= \rho^1 \cdot y^{1*} - w^{0*} \cdot \chi^0$$

where the last equality follows using (83). The middle term in the above decomposition is a theoretical measure of technical progress going from period 0 to 1. We can use the last equation in (85) to obtain an empirical expression for this theoretical measure, the *normalized Bennet indicator of technical progress*, $B_\tau(\rho^0,\rho^1,w^{0*},w^{1*},y^{0*},y^{1*},\chi^0,\chi^1)$:

$$(87)\ (1/2)[\Pi^1(\rho^0,\chi^1) - \Pi^0(\rho^0,\chi^1)] + (1/2)[\Pi^1(\rho^1,\chi^0) - \Pi^0(\rho^1,\chi^0)]$$
$$= \rho^1 \cdot y^{1*} - w^{0*} \cdot \chi^0 - (1/2)[y^{0*} + y^{1*}] \cdot [\rho^1 - \rho^0] - (1/2)[w^{0*} + w^{1*}] \cdot [\chi^1 - \chi^0]$$
$$= (1/2)[\rho^0 \cdot y^{1*} - w^{0*} \cdot \chi^1] - (1/2)[\rho^1 \cdot y^{0*} - w^{1*} \cdot \chi^0]$$
$$\equiv B_\tau(\rho^0,\rho^1,w^{0*},w^{1*},y^{0*},y^{1*},\chi^0,\chi^1).$$

Note that definition (87) is analogous to definition (18). Substituting (87) into (86) and making use of definitions (84) and (85) leads to the following *Bennet type exact decomposition of normalized value added growth* into explanatory components going from period 0 to 1 under our functional form assumptions:

$$(88)\ \Pi^1(\rho^1,\chi^1) - \Pi^0(\rho^0,\chi^0)$$

$$= B_\rho(\rho^0,\rho^1,y^{0*},y^{1*}) + B_\chi(\chi^0,\chi^1,w^{0*},w^{1*}) + B_\tau(\rho^0,\rho^1,w^{0*},w^{1*},y^{0*},y^{1*},\chi^0,\chi^1).$$

The above exact decomposition of the difference in normalized value added is a difference counterpart to the exact ratio decomposition of nominal value added that was obtained by Diewert and Morrison (1986) and Kohli (1990) where the translog value added function was the underlying functional form for the nominal value added function. However, the decomposition (88) is not as useful as these earlier decompositions for two reasons:

- It is somewhat complicated to go from (88) back to the nominal value added difference; i.e., we want a nice decomposition of $\Pi^1(p^1,x^1)–\Pi^0(p^0,x^0)$ whereas we have a nice decomposition of $\Pi^1(\rho^1,\chi^1) - \Pi^0(\rho^0,\chi^0)$;
- More fundamentally, the decomposition will depend on the analyst's choice of the α and β vectors and there is no clear rational for any particular specific choice.[33]

7. Conclusion

We have outlined three different approaches to the problem of decomposing the difference in a value added aggregate into explanatory components that are also differences. The third approach outlined in the previous section does not seem promising from an empirical point of view since different choices of the reference vectors α and β can lead to very different decompositions.

The first approach leads to very simple intuitively plausible decompositions but it has the disadvantage of being an approximate approach. It also assumes technical efficiency which is problematic when recessions occur as, for example, an inability to adequately adjust fixed capital inputs could result in inefficiencies in production. However, it is possible to reinterpret the first approach as an axiomatic approach where we choose the Bennet indicators of price and quantity change as being "best" from the viewpoint of the test approach.[34] The resulting productivity index combines the effects of technical progress and improvements in technical and allocative efficiency. Thus from the viewpoint of the test approach to indicators, one might view Approach 1 as "best".

The second approach seems "best" from the viewpoint of the economic approach to indicators. The drawback to the approach is that it is somewhat computationally intensive.

[33] This is the same problem that makes applications of directional distance function productivity studies problematic; there is no clear rational for any particular choice of the chosen direction and results are very much dependent on this choice. Diewert and Mizobuchi (2009; 368), in the context of a normalized quadratic cost function, noted that a reasonable standard choice for the weighting vector α is $\alpha \equiv y^0/p^0 \cdot y^0$. The vector of period t normalized prices, $\rho^t \equiv p^t/\alpha\cdot p^t$, can then be interpreted as a period t vector of real prices using a fixed-base Laspeyres price index to do the deflation of nominal prices. Even if this is taken as the appropriate choice of α, in our context of a normalized quadratic value added function, we still face the problem of making an appropriate choice for the β vector.

[34] See Diewert (2005) who developed a test approach for indicators and found that the Bennet indicators seemed "best" from the viewpoint of the test approach to indicators just as the Fisher index seems "best" from the viewpoint of the usual index number test approach.

It is also the case that our assumption that the actual period t production possibilities set can be well approximated by the free disposal conical hull of past observations may not be an accurate assumption. However, we like the fact that the approach is able to separate out the effects of technical progress and inefficiency, at least to some extent.

Finally, we recommend that Approaches 1 and 2 be implemented using prices that are deflated by a consumer price index or some other exogenous deflator that is suitable for the purpose at hand. This is particularly important for Approach 1 because the accuracy of the first order approximations used in this approach will be greatly improved by the removal of general inflation from the nominal prices.

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