The Aggregation of Capital over Vintages in a Model of Embodied Technical Progress

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Abstract

The paper considers how to measure capital in a model where technical progress is embodied in new units of capital. This embodiment model also assumes that once new units of capital are installed, it cannot be "unbolted" and sold on the second hand market. A significant difference between this Solow-Harper model and the traditional capital services model due to Jorgenson and his coworkers is that rising real wage rates will generally induce early retirement of assets; i.e., this model can provide an explanation for obsolescence. The paper studies how to aggregate over vintages and how to measure depreciation in the context of this embodiment model. These problems are more complicated than the corresponding problems in the traditional capital services model because the age of retirement of an asset is endogenous in the embodiment model. The paper uses duality theory to simplify the exposition.

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Keywords

Aggregation of capital, embodiment of technical progress, depreciation, deterioration, obsolescence, index number theory, the new goods problem, duality theory.

1. Introduction

Michael Harper (2007) has presented some serious criticisms of the traditional model for the aggregation of capital services over vintages of the same type of asset. The most important points in the *Harper critique* are as follows:

- The traditional vintage aggregation model is seriously flawed due to its neglect of induced obsolescence.

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- An alternative Solow (1960) type capital services model where technical progress is *embodied* in new units of the capital stock is more appropriate and reduces the procyclical volatility of the Solow residual.
- The traditional method of capital services aggregation neglects the effects of increasing real wages.

The primary purpose of the present paper is to present the main features of the Solow-Harper vintage model for the measurement and aggregation of capital services using duality theory, which simplifies the presentation considerably. A secondary purpose is to provide some guidance to economic statisticians on how this alternative model can be implemented. A final purpose is to provide some insights into the nature of obsolescence.

There are two main ideas that form the basis for this alternative model for measuring capital. The first idea is due to Solow (1960), who criticized the traditional method[2] for incorporating technical change into an aggregate production function as follows:

"It is an implication of (2)[3] that 'technical change' is peculiarly disembodied. It floats down from the outside. If K and L are held constant, equation (2) predicts that output will increase anyway, approximately exponentially at rate λ. If K and L both increase exponentially at rate γ, then output will simply increase at rate $\gamma + \lambda$. In particular, the pace of investment has no influence on the rate at which technique improves. It is as if all technical progress were something like time-and-motion study, a way of improving the organization and operation of inputs without reference to the nature of the inputs themselves. The striking assumption is that old and new capital equipment participate equally in technical change. This conflicts with the casual observation that many if not most innovations need to be embodied in new kinds of durable equipment before they can be made effective." Robert M. Solow (1960; 90-91).

Solow went on to postulate that technical progress occurred only by the creation of new improved units of capital equipment and this improved capital could be used by firms in a Cobb-Douglas (1928) production function context (with Hicks neutral technical progress) but once the new capital was purchased by a firm, no further technical progress could take place for that firm.

The second main set of ideas is due to Solow, Tobin, von Weizsäcker and Yaari (1966) and Harper (2007), who used the basic Solow (1960) framework but they relaxed the assumption that the vintage production functions must be Cobb-Douglas. They also relaxed the assumption that technical change must be Hicks neutral technical change; they allowed technical progress to be labor or capital augmenting or a mixture of the two types.[4] Under these relaxed assumptions, the decision to retire an asset becomes endogenous and depends upon the (rising) real wage rate:

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[2] The traditional method (which assumes disembodied Hicks neutral technical change) was in fact used by Solow in his earlier important contribution to modeling technical progress, Solow (1957).

[3] Equation (2) in Solow's paper is the aggregate production function equation $Q(t) = Be^{\lambda t} L(t)^{\alpha} K(t)^{1-\alpha}$, where Q(t), L(t) and K(t) are output, labor and capital at time t respectively. The form of technical progress is output augmenting or Hicks neutral technical progress.

[4] Solow, Tobin, von Weizsäcker and Yaari assumed that the production functions that used each new vintage of capital were Leontief (1941) whereas Harper did not restrict the functional form of the vintage production functions.

"One normal trend of technological progress will be a rising trend of the real wage rate. Since existing capital operates under fixed coefficients, there will eventually come a time in the life of every vintage investment when the wage costs of using it to produce a unit of output will exceed one unit of output. At that instant the investment may be said to have become obsolete as a result of the competition of more modern capital; it will be retired from production—permanently, unless the real wage should temporarily fall." R.M. Solow, J. Tobin, C.C. von Weizsäcker and M. Yaari (1966; 79).

The above authors spell out their assumptions on the vintage production functions and on how technical progress takes place in their model as follows:

"The model assumes fixed coefficient technology with embodied technical progress. Once capital has been put into place, there is no possibility of substituting capital for labor or vice versa; the output-capital and output-labor coefficients are fixed for the life of the capital. ... Technical progress consists of improvement in one or both of the output-input coefficients. But the improved coefficients apply only to new vintage capital, not to investments made in the past." R.M. Solow, J. Tobin, C.C. von Weizsäcker and M. Yaari (1966; 80).

The above authors go on to point out that their model can be used to study the economics of obsolescence.

The advantage of the Solow-Harper vintage capital model is that it relaxes the strong *separability assumptions* that are implicitly or explicitly assumed by the traditional vintage capital model; i.e., the traditional model assumes that the sequence of used asset prices and vintage user costs exist as entities that are independent of the actions of the firm with respect to other inputs and outputs and their prices.[5] However, dropping the traditional separability conditions comes at a cost: in order to implement this alternative embodiment model, it is necessary to have more information than the economic statistician may be able to collect.

An outline of the rest of the paper follows.

Section 2 provides an introduction to the mechanics of the Solow-Harper vintage machines model of production for an industry. In order to simplify the presentation, this paper discusses only the problems involved in aggregating over the vintages of a single type of capital.[6] In this introductory section, we assume that there is no technical progress but we focus on the valuation of each vintage of capital over time as it is used. The assumption is made that once a unit of new capital has been purchased, it has no alternative use; i.e., it is bolted down until it is retired and discarded by the purchasing firm. In order to accomplish the valuation of the vintage capital inputs, it will be necessary to make some assumptions about the nature of the vintage production functions and the pattern of future expected prices of industry outputs and other inputs. We show how aggregate estimates of (cross sectional) depreciation can be made and how aggregate

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[5] The traditional approach to the aggregation of capital is discussed by Christensen and Jorgenson (1969) (1973), Diewert (1980) (2005a) (2005b), Diewert and Lawrence (2000), Harper, Berndt and Wood (1989), Hulten (1990) (1996), Jorgenson (1989) (1996), Jorgenson and Griliches (1967) (1972), Schreyer (2007) and Solow (1957).
[6] Normal index number theory can be used to aggregate over different types (and vintages) of capital.

capital stocks and flows can be computed, where the aggregation takes place across vintages of the capital input.

Section 3 shows that the usual relationships between vintage user costs and vintage asset prices hold in this Solow-Harper machines model under certain assumptions.

Section 4 looks at the implications of the Solow-Harper machines model for cross sectional vintage depreciation versus the corresponding time series depreciation. The material in this section indicates that there will be a general tendency for the time series rates to be greater than their cross sectional counterparts.

Sections 5 and 6 sharpen some of the general results obtained in the previous 3 sections but at the cost of making more specific assumptions on the vintage production functions. In section 5, the vintage production functions are assumed to be Leontief while section 6 assumes Cobb Douglas functional forms.

Up to this point, it is assumed that the vintage capital inputs have geometric deterioration rates that are independent of the firm's actions. In section 7, this assumption is relaxed and it is assumed that as the capital input ages, an increasing amount of maintenance is required to keep the asset functioning properly.

Finally, sections 8 and 9 introduce technical progress into the vintage production functions. Section 8 assumes capital augmenting technical progress and it is found that the material in the previous sections can readily be adapted to this type of technical progress. Section 9 allows for both labor and capital augmenting technical change and it is found that labor augmenting technical progress is the key to offsetting the effects of rising real wage rates and the resulting shortening of asset lives.

Section 10 concludes the analytic part of the paper by considering how a capital services aggregate can be calculated when a new model of a capital input is used by an industry along with the old vintage capital inputs. It turns out that this problem is isomorphic to the general new goods problem and using this literature, two techniques for dealing with the problem are suggested and compared in a numerical example.

Section 11 concludes.

2. A Vintage Machines Model of Production with No Technological Progress

Consider a simple production model where newly produced units of capital K are combined with units of labour (or an aggregate of labour and intermediate inputs) L in order to produce units of output Y according to the following *production function* relationship:

(1) $Y = f(L,K)$.

We assume that the production function f is continuous, nondecreasing, concave and positively linearly homogeneous in its two arguments and is strictly positive if both inputs are positive.

Given that the firm faces the positive price p for units of output and the positive wage rate w,[7] the *unit profit function* for the firm, π(p,v), is defined as the maximum gross operating profit that the firm can make using one unit of capital:[8]

$$(2)\ \pi(p,w) \equiv \max_L \{pf(L,1) - wL\}.$$

Since the production function is subject to constant returns to scale, the *gross profit* or *operating surplus* for a firm that uses K new units of capital is π(p,w)K and so π(p,w) can be interpreted as the price that the firm is willing to pay for the use of one unit of new capital for one period if it faces prices p and w; i.e., π(p,w) is the *user benefit* or *imputed price of capital services* for one unit of newly installed capital.

We assume that once a new unit of capital is purchased and installed by a firm, it cannot be "unbolted" and sold in subsequent periods in the second hand market. It is this feature of the Solow model that differentiates it from the traditional capital services model that assumes the existence of second hand asset markets that can provide observable opportunity costs for the units of vintage capital in use by a firm.[9]

We allow for the fact that the productivity of a newly purchased unit of capital will generally *deteriorate* over time.[10] Thus we assume that the production function $f_n(L,K)$ that describes the maximum output that can be produced by L units of labour and K units of capital that were installed n periods ago can be defined in terms of the production function f given by (1), which describes the technology for newly purchased and installed units of capital, as follows:

$$(3)\ f_n(L,K) \equiv f(L,[1-\delta]^n K)\ ; \qquad n = 0,1,2, \ldots$$

[7] We use the term "wage rate" as a shorthand notation for the aggregate price of all inputs used by the industry.

[8] It can be shown that π(p,w) is nonnegative, positively linearly homogeneous and convex in p,w, nondecreasing in p and nonincreasing in w; see for example, Diewert (1973).

[9] It is this feature that makes the economic statistician's measurement problems very difficult since once the unit of capital has been purchased, it disappears into a black hole from the perspective of someone trying to observe and measure the contribution of the purchased capital to production in subsequent periods.

[10] Griliches described deterioration as follows: "The net stock concept is motivated by the observed fact that the value of a capital good declines with age (and/or use). This decline is due to several factors, the main ones being the decline in the life expectancy of the asset (it has fewer work years left), the decline in the physical productivity of the asset (it has poorer work years left) and the decline in the relative market return for the productivity of this asset due to the availability of better machines and other relative price changes (its remaining work years are worth less). One may label these three major forces as exhaustion, deterioration and obsolescence." Zvi Griliches (1963; 119). For additional material on the concept of deterioration and on terminology, see Triplett (1996).

where $0 \le \delta < 1$ is the *one period (geometric) deterioration rate*.[11] Note that the vintage production function f that applies to newly installed capital is equal to f_0 so that $f_0(L,K)$ equals f(L,K). Note also that the vintage production function that is applicable to units of capital K that were installed in the previous period is $f_1(L,K)$, which in turn equals $f(L,[1-\delta]K)$ and so on.

We now calculate the unit profit function $\pi_n(p,w)$ for a firm in this industry that is using one unit of capital that was installed n periods ago:

$$
\begin{aligned}
(4)\ \pi_n(p,w) &\equiv \max{}_L \{pf_n(L,1) - wL\} && n = 0,1,2, \ldots \\
&= \max{}_L \{pf[L,(1-\delta)^n 1] - wL\} && \text{using (3)} \\
&= \max{}_L \{p(1-\delta)^n f[L/(1-\delta)^n,1] - wL\} && \text{using the linear homogeneity of f} \\
&= \max{}_L \{p(1-\delta)^n f[L/(1-\delta)^n,1] - w(1-\delta)^n L/(1-\delta)^n\} \\
&= \max{}_{L^*} \{p(1-\delta)^n f[L^*,1] - w(1-\delta)^n L^*\} && \text{letting } L^* \equiv L/(1-\delta)^n \\
&= (1-\delta)^n \max{}_{L^*} \{pf[L^*,1] - wL^*\} \\
&= (1-\delta)^n \pi(p,w) && \text{using definition (2).}
\end{aligned}
$$

Thus the amount of gross profits earned today by a unit of capital installed n periods ago, $\pi_n(p,w)$, is equal to $(1-\delta)^n$ times the amount earned by a newly installed unit of capital, $\pi(p,w)$. Put another way, the value of the *capital services* yielded by a unit of capital that was installed n periods ago, $\pi_n(p,w)$, is equal to $(1-\delta)^n$ times the value of the capital services yielded by a newly installed unit of capital. This is an intuitively plausible result, given our assumptions about the nature of the vintage production functions.

We now turn our attention to the problem of determining the asset value for a newly installed unit of capital in period 0 and comparing this value to the period 0 asset values for capital installed in previous periods. We assume that a firm in the industry that purchases one unit of installed capital at the beginning of period 0 faces the output price $p^0 > 0$ and the variable input price $w^0 > 0$ in period 0. The firm forms expectations about the course of future output prices p^t and wage rates w^t. We also suppose that all firms in the industry face the *nominal opportunity cost of capital or interest rate* r at the beginning of period 0 and that this interest rate is expected to persist indefinitely.[12] With these assumptions, we can now determine the (expected) *value to the firm of purchasing one unit of newly installed capital* at the beginning of period 0, P_0^0:

$$
\begin{aligned}
(5)\ P_0^0 &\equiv \textstyle\sum_{n=0}^{\infty} \pi_n[p^n,w^n]/(1+r)^n \\
&= \textstyle\sum_{n=0}^{\infty} (1-\delta)^n \pi[p^n, w^n]/(1+r)^n && \text{using (4).}
\end{aligned}
$$

[11] If $\delta = 0$, then we have an infinite one hoss shay model. Solow, Tobin, von Weizsäcker and Yaari (1966; 81) use the term "one hoss shay". A finite lifetime one hoss shay model may be more appropriate for many infrastructure or sunk cost type investments rather than our geometric deterioration model.

[12] This assumption can readily be relaxed at the cost of some additional notational complexity.

Thus the asset value to the firm of a new unit of capital purchased at the beginning of period 0, P_0^0, is equal to the discounted expected value of the future gross profits, where the undiscounted expected period n profits are equal to $\pi_n[p^n,w^n] = (1-\delta)^n \pi[p^n, w^n]$.[13]

We now determine the *beginning of period 0 asset value of a unit of capital that was installed n periods ago*, P_n^0:

$$(6)\ P_n^0 \equiv \sum_{k=0}^{\infty} \pi_{n+k}[p^k, w^k]/(1+r)^k \qquad n = 1,2, \ldots$$
$$= \sum_{k=0}^{\infty} (1-\delta)^{n+k} \pi[p^k, w^k]/(1+r)^k \qquad \text{using (4)}$$
$$= (1-\delta)^n \sum_{k=0}^{\infty} (1-\delta)^k \pi[p^k, w^k]/(1+r)^k \qquad \text{rearranging terms}$$
$$= (1-\delta)^n P_0^0 \qquad \text{using (5).}$$

Thus the period 0 asset value to a firm that is using a unit of capital that is n periods old, P_n^0, is equal to one minus the deterioration rate raised to the power n, $(1-\delta)^n$, times the period 0 asset value to a firm that is using a newly installed unit of capital, P_0^0.

The decline in asset value due to a unit increase in age at the same point in time is known as the amount of *cross sectional depreciation* of the asset.[14] Thus the amount of cross sectional depreciation for one unit of an asset that is n periods old at the beginning of period 0, D_n^0, is defined as follows:

$$(7)\ D_n^0 \equiv P_n^0 - P_{n+1}^0 \qquad n = 0,1,2, \ldots$$
$$= (1-\delta)^n P_0^0 - (1-\delta)^{n+1} P_0^0 \qquad \text{using (6) twice}$$
$$= \delta(1-\delta)^n P_0^0 .$$

Using (7) for n = 0, we see that the cross sectional depreciation for a new asset at the beginning of period 0 is D_0^0, which is equal to the geometric deterioration rate δ times the period 0 asset price for a new unit of capital P_0^0. Using this fact, it can be seen that (7) implies that

$$(8)\ D_n^0 = (1-\delta)^n D_0^0 ; \qquad n = 1,2, \ldots .$$

[13] We have not spelled out the supply of capital part of the industry model. We assume that the demand price for a new unit of capital stock at the beginning of period 0 is given by (5) and suppliers take this price as given and produce however many units will maximize their period 0 profits. Thus demanders for new units of capital determine the price of a new unit of capital in this machines model and suppliers determine the quantity delivered of new machines. This is consistent with Solow's interpretation of his model: "If I(t) > 0, so that gross investment is actually occurring, the market value of a unit of brand new capital must also be equal to its cost of production." Robert M. Solow (1960; 100). However, if P_0^0 becomes too low, then suppliers will decide to supply 0 units of new capital to the industry. In the traditional separable capital approach, suppliers of new units of capital determine the price of a new machine and demanders determine the quantity delivered. A more general model would combine these demand side and supply side approaches.

[14] This terminology is due to Hill (1999) (2000) who distinguished the decline in second hand asset values due to aging (cross sectional depreciation) from the decline in an asset value over a period of time (time series depreciation). Triplett (1996; 98-99) uses the cross section definition of depreciation and shows that it is equal to the concept of capital consumption in the national accounts but he does this under the assumption of no expected real asset inflation. We will examine the relationship of cross section to time series depreciation in section 4 below.

We now consider the problems involved in forming *stock and flow aggregates for the capital used in the industry*. Suppose that K_0^0 new units of capital are purchased by the industry at the beginning of period 0 and that K_n^0 units of capital were purchased by the industry n periods ago, for n = 1,2, … . Suppose also that we tentatively define a *beginning of period 0 (net) capital stock aggregate*, K^0, as follows:

(9) $K^0 \equiv K_0^0 + (1-\delta)K_1^0 + (1-\delta)^2K_2^0 + \ldots$.

Now the beginning of period 0 asset value of the new units of capital is $P_0^0K_0^0$, the beginning of period 0 asset value of the one period old units of capital is $P_1^0K_1^0$, the beginning of period 0 asset value of the two period old units of capital is $P_2^0K_2^0$ and so on. Thus the *industry aggregate asset value for capital goods of all vintages at the beginning of period 0* will be:

$$\begin{aligned}(10)\ V_K^0 &\equiv P_0^0K_0^0 + P_1^0K_1^0 + P_2^0K_2^0 + \ldots \\ &= P_0^0K_0^0 + (1-\delta)P_0^0K_1^0 + (1-\delta)^2P_0^0K_2^0 + \ldots && \text{using (6)} \\ &= P_0^0\,[K_0^0 + (1-\delta)K_1^0 + (1-\delta)^2K_2^0 + \ldots\,] \\ &= P_0^0\,K^0 && \text{using (9).}\end{aligned}$$

Thus the aggregate asset value of capital over all vintages at the beginning of period 0, V_K^0, is equal to P_0^0, the price of a new unit of the capital stock at the beginning of period 0, times K^0, the geometric capital stock aggregate defined by (9) above. Thus aggregation over vintage stocks is very straightforward in this Solow-Harper vintage capital model.[15]

A similar aggregation result can be derived for the cross sectional amounts of depreciation, D_n^0, defined by (8) above. Define the *aggregate period 0 amount of cross sectional depreciation*, D^0, as the sum of the vintage amounts of cross sectional depreciation as follows:

$$\begin{aligned}(11)\ D^0 &\equiv D_0^0\,K_0^0 + D_1^0\,K_1^0 + D_2^0\,K_2^0 + \ldots \\ &= D_0^0\,K_0^0 + (1-\delta)D_0^0\,K_1^0 + (1-\delta)^2D_0^0\,K_2^0 + \ldots && \text{using (8)} \\ &= D_0^0\,[K_0^0 + (1-\delta)K_1^0 + (1-\delta)^2K_2^0 + \ldots\,] && \text{rearranging terms} \\ &= D_0^0\,K^0 && \text{using definition (9)} \\ &= \delta\,P_0^0\,K^0 && \text{using (7) for n = 0.}\end{aligned}$$

Thus the aggregate amount of cross sectional depreciation in period 0, D^0, is equal to the one period deterioration rate δ times the price of a new asset in period 0, P_0^0, times the beginning of period 0 aggregate capital stock K^0.

[15] The key to this aggregation result is equations (6), where it can be seen that all of the vintage asset prices vary in a strictly proportional manner to the price of a new asset. Thus we can apply Hicks' (1939; 312-313) aggregation theorem; i.e., if all prices in the aggregate move in strict proportion over time, then any one of these prices can be taken as the price of the aggregate. The corresponding quantity aggregate is equal to the value aggregate divided by the chosen price.

A final aggregation result can be derived for capital services. The period 0 *capital service charge* or *imputed rental price*[16] or *user charge* for a unit of capital that is n periods old at the start of period 0, u_n^0, can be defined as the gross income that one unit of the vintage asset can earn in period 0, $\pi_n(p^0,w^0)$:

$$(12)\ u_n^0 \equiv \pi_n(p^0,w^0)\ ; \qquad n = 0,1,2,\ldots$$
$$= (1-\delta)^n\, \pi(p^0,w^0) \qquad \text{using (4).}$$

Using (12), it can be seen that all of the period 0 vintage user charges vary proportionally[17] to the user charge for a new asset, u_0^0; i.e., we have:

$$(13)\ u_n^0 = (1-\delta)^n\, u_0^0\ ; \qquad n = 1,2, \ldots .$$

Define the *aggregate value of capital services in period 0*, V_{KS}^0, as the sum over all vintages of the values of the vintage user charges in period 0:

$$(14)\ V_{KS}^0 \equiv u_0^0 K_0^0 + u_1^0 K_1^0 + u_2^0 K_2^0 + \ldots$$
$$= u_0^0 K_0^0 + (1-\delta)\, u_0^0 K_1^0 + (1-\delta)^2 u_0^0 K_2^0 + \ldots \qquad \text{using (13)}$$
$$= u_0^0 K^0 \qquad \text{using (9).}$$

Thus the period 0 aggregate value of capital services over all vintages, V_{KS}^0, is equal to u_0^0, the value of the capital services of a new unit of the capital stock at the beginning of period 0, times K^0, the geometric capital stock aggregate defined by (9) above.

At this point, the reader should notice the similarity of this vintage machines model (which does not assume separability of the capital aggregate from other inputs and output) to the separable geometric depreciation model of vintage capital services that has been utilized by Jorgenson (1989) (1996) and his coworkers.[18] In fact, at this point, we might conclude that the two models are identical in their implications for statistical agency measurement practices.

However, there are some important differences between the two approaches. In the Jorgensonian geometric model of depreciation, the investment price of a new unit of capital (the counterpart to our asset value for a newly installed unit of capital P_0^0) is taken to be an exogenous variable (and nothing is known about how this price will trend over time) whereas in our approach, P_0^0 is an endogenous variable, which is determined by the nonseparable vintage production functions and expectations about the future course of output and input prices and interest rates. Thus in the (realistic) case where variable input prices are expected to increase more rapidly than output prices in the industry, at some future period T, the user benefit of a new unit of capital in period T, equal to $u_0^T \equiv$

[16]Since a newly purchased machine is bolted down by our assumptions, its services cannot be rented out to other firms and hence the rental price is an imputed one (which is equal to the period 0 gross profits that the new machine can earn in period 0).
[17] Thus again, Hicks' (1939; 312-313) aggregation theorem is applicable to the capital services aggregation problem.
[18] See Jorgenson and Griliches (1967) (1972) and Christensen and Jorgenson (1969) (1973) among others.

$\pi(p^T,w^T)$, will fall to zero and the industry will shut down.[19] Even before this point is reached, as t approaches T, the asset value for a new unit of capital P_0^t will probably fall below the cost of producing the asset and, while the industry will not shut down immediately, no new units of capital will be purchased from suppliers of the new asset so a gradual decline in the industry will take place. This gloomy industry outlook can be offset by certain types of technical progress, a topic that we will address in later sections.

There is another important difference between the vintage machines model and the separable geometric depreciation model. Suppose that technology is such that in period T in the future, $\pi(p^T,w^T)$ falls to zero and the industry shuts down. Suppose that prices in period T are perfectly anticipated in period 0. Then a new asset purchased at the beginning of period 0 has an expected life of T+1 periods. However, when a new asset is purchased at the beginning of period 1, it has an expected life of only T periods.[20] Thus rising real wage rates in this simplified no technical progress version of the Solow-Harper machines model lead to the presumption that the expected asset life of a new asset is *not constant* and in fact, is declining.

In the following section, we show how the sequence of period 0 vintage capital user charges is related to the sequence of vintage asset prices. This relationship is very useful in the traditional model of vintage capital services because it leads to an easy formula for a user cost or capital service price in terms of potentially observable used asset prices. A similar result can be derived in the present context for a *new* asset under the assumption that expectations made during periods 0 and 1 do not change.

3. The Relationship between User Costs or Charges and Asset Prices

Recall definitions (6), which defined the beginning of period 0 asset values for units of capital that were installed n periods ago, P_n^0. Now let us move forward one period and define the sequence *beginning of period 1 asset values for units of capital that were installed n periods ago*, P_n^1:

(15) $P_n^1 \equiv \sum_{k=0}^{\infty} \pi_{n+k}[p^{k+1},w^{k+1}]/(1+r)^k$ $\qquad n = 0,1,2, \ldots$

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[19] At least, the industry will shut down for most technologies. If the vintage technologies are Cobb Douglas, then the industry will never shut down; see section 6 below. Harper (2007) explicitly noted this fact and it is implicit in Solow (1960).

[20] Harper quite properly regarded this as an obsolescence effect: "Faced with a wage increase, the firm will reduce the amount of labor and output slightly, raising average labor productivity, consistent with what Cooper and Haltiwanger (1993) have observed happening to plants as they aged. In the long run, technological improvements generally lead to investments in improved capital goods which, in turn, bid for scarce labor, driving a persistent upward rotation in the ray representing revenue equals cost. The effect of *obsolescence* is just the rent lost due to the persistent rise in wages relative to the price of output." Michael J. Harper (2007). Solow, Tobin, Weizsäcker and Yaari (1966; 79) also described the induced closing of old plants due to rising real wage rates as an obsolescence effect in their model: "One normal consequence of technological progress will be a rising trend of the real wage rate. Since existing capital operates under fixed coefficients, there will eventually come a time in the life of every vintage of investment when the wage costs of using it to produce a unit of output will exceed one unit of output. At that instant the investment may be said to have become obsolete as the result of the competition of more modern capital; it will be retired from production—permanently, unless the real wage should temporarily fall."

$= \sum_{k=0}^{\infty} (1-\delta)^{n+k} \pi[p^{k+1},w^{k+1}]/(1+r)^k$ using (4)
$= (1-\delta)^n \sum_{k=0}^{\infty} (1-\delta)^k \pi[p^{k+1},w^{k+1}]/(1+r)^k$ rearranging terms.

A close examination of (15) shows that all of the period 1 vintage asset values are proportional to the period 1 price of a new asset, P_0^1; i.e., we have the following counterpart to equations (6)

$$(16)\ P_n^1 = (1-\delta)^n P_0^1 \qquad n = 1,2, \ldots .$$

Now use (15) for n = 0 and we have the following expression for the asset price of a new unit of capital at the beginning of period 1:

$$(17)\ P_0^1 = \pi[p^1,w^1] + (1-\delta)\,\pi[p^2,w^2]/(1+r)^1 + (1-\delta)^2\,\pi[p^3,w^3]/(1+r)^2 + \ldots .$$

Now *suppose that the price expectations formed at the beginning of period 0 continue to hold at the beginning of period 1*. Using definition (6) for the beginning of period 0 price for a new machine, P_0^0, we have:

$$(18)\ P_0^0 = \pi[p^0,w^0] + (1-\delta)\,\pi[p^1,w^1]/(1+r) + (1-\delta)^2\,\pi[p^2,w^2]/(1+r)^2 + \ldots$$
$$= u_0^0 + (1-\delta)\,\pi[p^1,w^1]/(1+r) + (1-\delta)^2\,\pi[p^2,w^2]/(1+r)^2 + \ldots \qquad \text{using (12) for } n = 0$$
$$= u_0^0 + [(1-\delta)/(1+r)]\{\pi[p^1,w^1] + (1-\delta)\,\pi[p^2,w^2]/(1+r) + \ldots\}$$
$$= u_0^0 + [(1-\delta)/(1+r)]\,P_0^1 \qquad \text{using (17).}$$

Equation (18) can be used to solve for the user cost[21] of a newly installed machine at the beginning of period 0, u_0^0, in terms of the beginning of period 0 and 1 asset prices for a new machine, P_0^0 and P_0^1 respectively:[22]

$$(19)\ u_0^0 = P_0^0 - (1-\delta)P_0^1/(1+r) = (1+r)^{-1}[rP_0^0 + \delta P_0^1 - (P_0^1 - P_0^0)].$$

This is one form of the "traditional" formula for the user cost of a new unit of capital.[23] Once the period 0 user charge for a new unit of capital has been determined by (19), the period 0 user charges for earlier purchases of the capital good, u_n^0, are equal to $(1-\delta)^n u_0^0$

---

[21] We again remind the reader that in the present model, it would be more accurate to call u_0^0 a *user benefit* or *user charge* rather than a user cost. Note also that u_0^0 is equal to $\pi(p^0,w^0)$, the first period cash flow earned by a newly installed unit of capital at the beginning of period 0.

[22] Note that we are discounting cash flows to the beginning of period 0 and thus u_0^0 can be interpreted as a *beginning of the period user charge*. Multiplying u_0^0 by (1+r) leads to an *end of period user charge*, which is more consistent with accounting conventions; e.g., see Peasnell (1981; 56). These discounting conventions are discussed in more detail in Diewert (2005a) (2005b; 8).

[23] Using the traditional separable model of capital, variants of equation (19) were derived by Christensen and Jorgenson (1969; 302) (1973), Jorgenson (1989; 10), Hulten (1990; 128), Diewert and Lawrence (2000; 276) and Diewert (1980; 470) (2005a) (2005b). Irving Fisher (1908; 32-33) derived these equations in words as follows: "Putting the principle in its most general form, we may say that for any arbitrary interval of time, the value of the capital at its beginning is the discounted value of two elements: (1) the actual income accruing within that interval, and (2) the value of the capital at the close of the period." Note that cash flows are discounted to the beginning of period 0 in formula (19) and so u_0^0 is termed a beginning of the period user cost or charge.

using equations (13). Thus if new units of the capital good are purchased in both periods 0 and 1 and expectations about future prices do not change over the course of period 0, then the sequence of period 0 vintage user costs, u_n^0, can be derived from a knowledge of r (the nominal interest rate facing producers in the industry), δ (the one period deterioration rate), and the period 0 and 1 asset prices for a new machine, P_0^0 and P_0^1. However, here is where we encounter a problem that is not a problem in the traditional vintage capital services model. The problem is this: in the Solow-Harper model, *there is no directly observable sequence of used asset prices that can be used in order to determine the deterioration rate* δ.[24] An alternative method for the determination of the deterioration rate in the Solow-Harper framework would be to look at the gross profits earned by firms in the same industry using different vintages of the capital input.[25]

In the following section, we will analyze the implications of the vintage machines model for determining whether time series depreciation differs from cross sectional depreciation.

4. Cross Sectional versus Time Series Depreciation

We begin this section with a definition of the *time series depreciation* of an asset. Define the *ex ante time series depreciation* of an asset that is n periods old at the beginning of period 0, Δ_n^0, to be its price at the beginning of period 0, P_n^0, less its expected price at the end of period 0, P_{n+1}^1; i.e.,

$$(20)\ \Delta_n^0 \equiv P_n^0 - P_{n+1}^1 \quad ; \qquad n = 0,1,2,\ldots$$

Time series depreciation is more important than cross sectional depreciation for national income accountants and for firms in regulated industries, because it is time series depreciation which appears in the System of National Accounts (to relate gross product to net product) and it is time series depreciation which determines the Regulatory Asset Base for the regulated firm. In this section, we will attempt to relate time series depreciation Δ_n^0 for an n period old asset in period 0 to its cross sectional counterpart defined by (7), $D_n^0 \equiv P_n^0 - P_{n+1}^0$.

A problem with (20) is that time series depreciation captures the effects of changes in *two* things: changes in *time* (this is the change in t from 0 to 1) and changes in the *age* of the asset (this is the change in n to n+1). Thus time series depreciation aggregates together two effects: the asset specific price change that occurred between time 0 and time 1 *and*

---

[24] It is the fact that used asset prices can be observed which enables us to determine δ in the traditional capital services model; e.g., see Hulten and Wykoff (1981a) (1981b). However, engineers may be able to provide an estimate for the deterioration rate δ.

[25] Alternatively, the sequence of gross profits earned by a vintage plant along with the corresponding sequence of variable inputs and outputs could be used to estimate the parameters of the atemporal production function f(L,K) characterizing the plant along with the deterioration rate δ. In principle, this estimation of the production function approach could be used in Solow type models where there are many different types of capital (whereas the approach suggested in the text would fail in situations where firms are using many types of capital).

the effects of asset aging (deterioration). The above definition of time series depreciation is the original definition of depreciation and it extends back to the very early beginnings of accounting theory.[26]

Using definition (20) for n = 0, we have:

$$
\begin{aligned}
(21)\ \Delta_0^{\,0} &\equiv P_0^{\,0} - P_1^{\,1} \\
&= P_0^{\,0} - (1-\delta)\, P_0^{\,1} && \text{using (16) for } n = 1 \\
&= \delta P_0^{\,1} - (P_0^{\,1} - P_0^{\,0}) \\
&= D_0^{\,0} - (1-\delta)(P_0^{\,1} - P_0^{\,0}) && \text{using (7) for } n = 0.
\end{aligned}
$$

Thus expected time series depreciation for a new asset at the beginning of period 0, $\Delta_0^{\,0}$, is equal to cross sectional depreciation for a new asset at the beginning of period 0, $D_0^{\,0}$, less one minus the deterioration rate δ times the anticipated price change over period 0 for a new asset of this type, $P_0^{\,1} - P_0^{\,0}$. Thus if the asset is expected to appreciate in price over the course of period 0, time series appreciation will be *less* than cross sectional depreciation whereas an expected fall in the asset price means that $\Delta_0^{\,0}$ will be *greater* than $D_0^{\,0}$, assuming that δ is between 0 and 1. Usually, $P_0^{\,1}$ will be less than $P_0^{\,0}$ (due to the effects of increasing real wage rates, which will cause the useful life of a new asset to be less at the end of period 0 as compared to its beginning of period useful life) so that typically, time series depreciation will be *larger* than its cross sectional counterpart in the Solow Harper model.

The third line in (21) shows that the period 0 time series depreciation for a new asset, $\Delta_0^{\,0}$, is equal to $\delta P_0^{\,1} - (P_0^{\,1} - P_0^{\,0})$, where the first term, $\delta P_0^{\,1}$, is a measure of cross sectional depreciation for the asset, but taken at the asset price prevailing at the end of period 0, $P_0^{\,1}$, instead of the beginning of period 0 asset price $P_0^{\,0}$, and the second term, $-(P_0^{\,1} - P_0^{\,0})$, subtracts the expected period 0 price appreciation of the asset over period 0 from the cross sectional depreciation term. Now recall formula (19) for the beginning of period 0 user charge for the use of a new unit of the asset, $u_0^{\,0}$. We can switch to an *end of period user charge* $U_0^{\,0}$ by multiplying $u_0^{\,0}$ by (1+r) and the resulting formula is:

$$(22)\ U_0^{\,0} \equiv (1+r)u_0^{\,0} = [rP_0^{\,0} + \delta P_0^{\,1} - (P_0^{\,1} - P_0^{\,0})] = rP_0^{\,0} + \Delta_0^{\,0}$$

where the last equality follows using the third line in (21). Thus the end of period user charge, $U_0^{\,0}$, is equal to period 0 *waiting costs*, $rP_0^{\,0}$, plus *time series depreciation*, $\Delta_0^{\,0}$.

There is an important implication that falls out of the above algebra: in the Solow-Harper model of capital, *time series depreciation should include the effects of anticipated asset price change*; i.e., the term $-(P_0^{\,1} - P_0^{\,0})$ should be included in time series depreciation in addition to the deterioration term $\delta P_0^{\,1}$. This conclusion follows the advice given by P. Hill (2000; 6) and Hill and Hill (2003; 617)[27], who argued that a form of time series depreciation that included expected asset price change was to be preferred over cross

[26] See for example Matheson (1910; 35) and Hotelling (1925; 341).
[27] Hill and Hill assumed that there was no general inflation in their exposition.

sectional depreciation for national accounts purposes. In the usual capital services model, the inclusion of the expected asset price revaluation term, $-(P_0^1 - P_0^0)$, as a part of time series depreciation is controversial: some economists favor its exclusion from depreciation (these economists take a physical maintenance of capital position) and some favor its inclusion (these economists take a financial maintenance of capital position). The controversy dates back to Pigou (1941; 273-274), who favored a maintenance of physical capital approach and Hayek (1941; 276-277), who favored a maintenance of real financial capital approach.[28] However, in the context of our version of the Solow-Harper embodiment model, it seems clear that the anticipated price revaluation term the term should be included in time series depreciation.

Equations (21) related period 0 time series depreciation for a new asset to its cross sectional counterpart. It is possible to generalize (21) to the case of an asset that is n periods old at the start of period 0. From definitions (20), the time series depreciation for an asset of this type is:

$$
\begin{aligned}
(23)\ \Delta_n^0 &\equiv P_n^0 - P_{n+1}^1\ ; && n = 1,2,\ldots \\
&= (1-\delta)^n P_0^0 - (1-\delta)^{n+1} P_0^1 && \text{using (6) and (16)} \\
&= (1-\delta)^n \Delta_0^0 && \text{using line 2 in (21)} \\
&= (1-\delta)^n[D_0^0 - (1-\delta)(P_0^1 - P_0^0)] && \text{using line 4 in (21)} \\
&= D_n^0 - (1-\delta)^{n+1}(P_0^1 - P_0^0) && \text{using (8).}
\end{aligned}
$$

As mentioned above, normally, the discounted cash flow that a new asset is expected to generate at the start of period 0 will typically be greater than the discounted cash flow that a new asset is expected to generate at the end of the period, so that $P_0^1 - P_0^0$ will usually be negative. In this case, the last line in (23) shows that the time series depreciation of an asset that is n periods old at the beginning of period 0, Δ_n^0, will be *greater* than the counterpart cross sectional depreciation, D_n^0.

There is one more aspect of time series depreciation for an asset that cannot be resold during its useful life that needs to be discussed and that is whether time series depreciation for such an asset is uniquely determined. Recall equation (5), which equated the purchase price of a new asset at the beginning of period 0, P_0^0, to the discounted stream of profits that the asset is expected to yield over its useful life, $\sum_{t=0}^{\infty} \pi_t[p^t,w^t]/(1+r)^t$. Thus far in this paper, we have suggested that a firm purchasing one unit of a new asset at the beginning of period 0 should set a period t user charge for the use of the bolted down capital equal to the period t cash flow generated by the asset, $\pi_t[p^t,w^t]$. However, from the viewpoint of the purchasing firm, it does not seem necessary at first sight to make this particular allocation of user charges. Thus suppose that the firm decides to make a period t user charge equal to $c^t \geq 0$ for $t = 0,1,2,\ldots$, and these user charges satisfy the following equation:

$$(24)\ P_0^0 \equiv \sum_{t=0}^{\infty} c^t/(1+r)^t.$$

[28] For a more detailed discussion on these alternative income concepts, see Diewert (2006). Including the expected revaluation term in depreciation reflects Hayek's point of view but not Pigou's.

Then it can be shown that the firm will fully recover its purchase cost of the asset in a present value sense, provided that the sequence of user charges c^t satisfy equation (24).[29] Thus *it would seem that depreciation is largely arbitrary in a model where capital is bolted down or where costs are sunk*.[30] However, this arbitrariness can be removed if we note that at any point in time, the firm could be sold to other investors. These investors will value the existing bolted down capital stocks of the firm by the discounted value of the profits that are expected in future periods and thus accounting values for the firm's capital stocks should recognize this reality. This means that the c^t in (24) are no longer arbitrary; they should be set equal to the corresponding cash flows, $\pi_t[p^t,w^t]$, and we are back to the algebra already developed.[31]

We turn now to some concrete examples of the above general theory.

5. Leontief Technologies

In this section, we assume that the production function (1) that applies to units of new capital has the following *Leontief* (1941) *functional form*:[32]

$$(25)\ Y = f(L,K) \equiv \min\ \{L/a_L, K/a_K\}$$

where $a_L > 0$ and $a_K > 0$ are the input-output coefficients for labour and capital respectively. The unit capital profit function that corresponds to this production function is the following one:

---

[29] Using the chosen sequence of user charges satisfying (24), the beginning of period t undepreciated asset value would be $P_t^{t*} \equiv \sum_{k=t}^{\infty} c^k/(1+r)^{k-t}$ and the period t time series depreciation would be $\Delta_t^{t*} \equiv P_t^{t*} - P_{t+1}^{t+1*}$.

[30] This is the conclusion reached by Peasnell and Schmalensee as is indicated in the following quotations: "The concept of income employed in equation (4), $P_t - i_t A_{t-1}$, is very general. Profit P_t and asset book value A_{t-1} are not restricted to a particular accounting model, for example, to economic income. The only restrictions are those concerning the valuation of the opening and closing capital stocks, A_0 and A_N respectively. Opening book capital must be valued at outlay (i.e. $A_0 = C_0$); closing book capital must be valued at the amount expected to be received from disposal of the asset(s) (i.e. $A_N = R_N$). How capital stock is valued in the periods in between (t = 2, ... , N−1) is of no consequence whatsoever. Interim capital stocks, $A_1, A_2, \ldots, A_{N-1}$, can be valued at economic value, HC, RC, VO, or NRV: from an investment planning viewpoint, the choice is immaterial." K.V. Peasnell (1981; 54).
"It is important to recognize that the invariance Proposition does not imply that all depreciation schedules are equally socially desirable. Inappropriate choice of depreciation policy can lead to an intertemporal pattern of utility rates that bears no relation to the corresponding intertemporal pattern of capital costs. But the Proposition indicates that depreciation policy can be altered to produce more efficient rates without being unfair in a present value sense to utilities or their customers". Richard Schmalensee (1989; 295-296).

[31] If the purchase cost of a sunk cost type asset is below the discounted stream of cash flows that the asset is expected to generate, then we should make the accounting charges c^t proportional to the corresponding expected cash flow; see section 12 of Diewert (2005a) for the details.

[32] This is the functional form used by Solow, Tobin, Weizsäcker and Yaari (1966; 80) in their machines model: "The model assumes fixed coefficient technology with embodied technical progress. Once capital has been put into place, there is no possibility of substituting capital for labor or vice versa; the output-capital and the output-labor coefficients are fixed for the life of the capital." The model developed in this section is simpler than their model because we have not yet introduced embodied technical progress.

(26) $\pi(p,w) \equiv \max_L \{pf(L,1) - wL\} = pa_K^{-1} - wa_La_K^{-1}$.

Choose units of measurement for labour and capital so that $a_L = a_K = 1$. Assume also that in period 0, the profits generated by one unit of newly installed capital are positive so that

(27) $\pi_0(p^0,w^0) \equiv \pi(p^0,w^0) = p^0 - w^0 > 0$

where $p^0 > 0$ is the period 0 industry price of output and $w^0 > 0$ is the period 0 industry wage rate.

For simplicity, we assume that all firms in the industry expect output prices to grow at the geometric rate $1+i_Y > 0$ and wage rates to grow at the geometric rate $1+i_L > 0$. Thus we assume that:

(28) $p^t = (1+i_Y)^t p^0$; $w^t = (1+i_L)^t w^0$; $t = 1,2, \ldots$.

In order to work out the sequence of used asset prices and the sequences of cross sectional and time series depreciation rates for this model of production, it is necessary to consider two separate cases: one case where expected wage growth rate i_L is equal to or less than expected output price growth rate i_Y (this case is less interesting from an empirical point of view) and the case where wages are expected to grow faster than output prices.

Case (i): Output prices grow at least as fast as wages

For this case, we assume:

(29) $i_Y \geq i_L > 0$; $1 > (1-\delta)(1+i_Y)/(1+i_L) > 0$; $1 > \delta \geq 0$.

Using (2)-(5) and (26)-(29) it can be shown that the asset price for a newly installed unit of capital at the beginning of period 0 is

(30) $P_0^0 = \sum_{n=0}^{\infty} (1-\delta)^n \pi[(1+i_Y)^n p^0, (1+i_L)^n w^0]/(1+r)^n$
$= \sum_{n=0}^{\infty} (1-\delta)^n [(1+i_Y)^n p^0 - (1+i_L)^n w^0]/(1+r)^n$
$= p^0 A - w^0 B > 0$

where A and B are defined as follows:

(31) $A \equiv (1+r)/[(1+r) - (1-\delta)(1+i_Y)] > 0$; $B \equiv (1+r)/[(1+r) - (1-\delta)(1+i_L)] > 0$.

Once the price of a new asset P_0^0 has been determined using (30), equations (6) and (7) can be used to determine the sequence of period 0 vintage asset prices P_n^0 and cross sectional depreciation amounts D_n^0 for n = 0,1,2, ... In order to determine whether the cross sectional depreciation for an n period old asset at the beginning of period 0, D_n^0, is greater or less than the corresponding real time series depreciation, Δ_n^0, it is first necessary to calculate the price of a new asset at the beginning of period 1, P_0^1:

$$(32)\ P_0^{\,1} = \sum_{n=0}^{\infty} (1-\delta)^n\, \pi[(1+i_Y)^{n+1}p^0, (1+i_L)^{n+1}w^0]/(1+r)^n$$
$$> \sum_{n=0}^{\infty} (1-\delta)^n\, \pi[(1+i_Y)^{n}p^0, (1+i_L)^{n}w^0]/(1+r)^n \qquad \text{using (29)}$$
$$= P_0^{\,0} \qquad \text{using (30).}$$

Using (21), (23) and (32), we conclude that in this case, the sequence of vintage time series depreciation amounts, Δ_n^0, is strictly *less* than the corresponding cross sectional depreciation amounts, D_n^0. This is intuitively plausible since business conditions become better (in nominal terms at least) for each firm in the industry as time marches on due to the fact that the output price is growing at least as quickly as the wage rate. Thus each unit of capital employed in the industry makes a nominal capital gain as time moves forward.

Case (ii): The wage rate grows faster than the output price

For this case, we assume:

$$(33)\ i_L > i_Y \geq 0\ ; \quad 1 > \delta \geq 0\ ; \quad 1+r > 0.$$

This case is much more likely to hold than the previous case. The faster growth of wages compared to output prices means that, eventually, gross profits are driven to zero. Thus let T be the period where this happens; i.e., let $T \geq 1$ be such that:

$$(34)\ (1+i_Y)^T p^0 - (1+i_L)^T w^0 \geq 0 \text{ and}$$
$$(35)\ (1+i_Y)^{T+1} p^0 - (1+i_L)^{T+1} w^0 < 0.$$

Thus in this case, the value of a newly purchased unit of capital, $P_0^{\,0}$, is equal to a sum of at most T+1 terms:

$$(36)\ P_0^{\,0} = \sum_{n=0}^{T} (1-\delta)^n\, [(1+i_Y)^n p^0 - (1+i_L)^n w^0]/(1+r)^n$$

since $\pi[(1+i_Y)^n p^0, (1+i_L)^n w^0] = 0$ for $n > T$ and so the infinite summation in (5) reduces to a finite one.

Once the price of a new asset $P_0^{\,0}$ has been determined using (36), equations (6) and (7) can be used to determine the sequence of period 0 vintage asset prices $P_n^{\,0}$ and cross sectional depreciation amounts D_n^0 for n = 0,1,2, ... In order to determine whether the cross sectional depreciation for an n period old asset at the beginning of period 0, D_n^0, is greater or less than the corresponding real time series depreciation, Δ_n^0, we calculate $P_0^{\,1}$:

$$(37)\ P_0^{\,1} = \sum_{n=0}^{\infty} (1-\delta)^n\, \pi[(1+i_Y)^{n+1}p^0, (1+i_L)^{n+1}w^0]/(1+r)^n$$
$$= \sum_{n=0}^{T-1} (1-\delta)^n\, [(1+i_Y)^{n+1}p^0 - (1+i_L)^{n+1} w^0]/(1+r)^n$$
$$< (1+i_L)\sum_{n=0}^{T-1} (1-\delta)^n\, [(1+i_Y)^{n}p^0 - (1+i_L)^{n} w^0]/(1+r)^n \qquad \text{using } i_Y < i_L$$
$$\leq \sum_{n=0}^{T-1} (1-\delta)^n\, [(1+i_Y)^{n}p^0 - (1+i_L)^{n} w^0]/(1+r)^n \qquad \text{using } i_L \geq 0$$
$$\leq \sum_{n=0}^{T} (1-\delta)^n\, [(1+i_Y)^{n}p^0 - (1+i_L)^{n} w^0]/(1+r)^n \qquad \text{using (34)}$$

$= P_0^0$ using (36).

Using (21), (23) and (37), we have

(38) $\Delta_n^0 > D_n^0$ for $n = 0,1,2, ...$

Thus in this case where the wage rate is increasing more rapidly than the industry's output price, *time series depreciation*, Δ_n^0, *is strictly greater than the corresponding cross sectional depreciation*, D_n^0, for all vintages of capital in use at the beginning of period 0. Moreover, under the assumption that producers' expectations about future industry prices are realized, *each vintage of capital will have its expected length of life until retirement reduced by one period as each period expires*; i.e., in this rising real wage rate case, the industry is marching steadily towards extinction!

We turn now to an example where the vintage production functions are Cobb Douglas.

6. Cobb-Douglas Technologies

In this section, we assume that the production function (1) that applies to units of new capital has the following *Cobb Douglas* (1928) *functional form*:[33]

(39) $Y = f(L,K) \equiv (3/2)L^{2/3}K^{1/3}$.

The unit capital profit function (for a new unit of capital) that corresponds to this production function is the following one:

(40) $\pi(p,w) \equiv \max_L \{pf(L,1) - wL\} = \max_L \{p(3/2)L^{2/3} - wL\}$.

The solution to the above maximization problem is:

(41) $L(p,w) \equiv (p/w)^3$; $Y(p,w) \equiv (3/2)(p/w)^2$; $\pi(p,w) \equiv (1/2)p^3/w^2$.

Using (5) and (41), it can be shown that the asset price for a newly installed unit of capital at the beginning of period 0 is

$$\begin{aligned}(42)\ P_0^0 &= \textstyle\sum_{n=0}^{\infty} (1-\delta)^n \pi[(1+i_Y)^n p^0, (1+i_L)^n w^0]/(1+r)^n \\ &= (1/2)\,[p^0]^3\,[w^0]^{-2} \textstyle\sum_{n=0}^{\infty} [(1-\delta)(1+i_Y)^3(1+i_L)^{-2}(1+r)^{-1}]^n \\ &= \pi_0(p^0,w^0)\,/(1-\theta)\end{aligned}$$

where the parameter θ is defined as follows:

(43) $\theta \equiv (1-\delta)(1+i_Y)^3/(1+i_L)^2(1+r)$.

For this production model, we make the following assumptions:

[33] A labor share of 2/3 and a capital share of 1/3 is approximately valid for many industries..

(44) $0 < \theta < 1$; $0 < 1+i_Y$; $0 < 1+i_L$; $0 \leq \delta < 1$.

As noted by Harper (2007), when the vintage production functions are Cobb Douglas, old units of capital are never retired. Once the price of a new asset P_0^0 has been determined using (42), the general equations (6) and (7) can be used to determine the sequence of period 0 vintage asset prices P_n^0 (for assets that are n periods old at the start of period 0) and the corresponding cross sectional depreciation amounts D_n^0. In order to determine whether the cross sectional depreciation for an n period old asset at the beginning of period 0, D_n^0, is greater or less than the corresponding real time series depreciation, Δ_n^0, we first calculate the price of a new asset at the start of period 1, P_0^1:

$$\begin{aligned}(45)\ P_0^1 &= \textstyle\sum_{n=0}^{\infty} (1-\delta)^n \pi[(1+i_Y)^{n+1}p^0, (1+i_L)^{n+1}w^0]/(1+r)^n \\ &= (1/2)\,[p^0]^3\,[w^0]^{-2}\,(1+i_Y)^3(1+i_L)^{-2}\textstyle\sum_{n=0}^{\infty} [(1-\delta)(1+i_Y)^3(1+i_L)^{-2}(1+r)^{-1}]^n \\ &= (1+i_Y)^3(1+i_L)^{-2}\,P_0^0 \qquad \text{using (42).}\end{aligned}$$

From (45), it can be seen that

$$(46)\ P_0^0 > P_0^1 \text{ if and only if } (1+i_L) > (1+i_Y)^{3/2}.$$

Using (21), (23) and (46), we have

$$(47)\ \Delta_n^0 > D_n^0 \text{ if and only if } (1+i_L) > (1+i_Y)^{3/2} \qquad \text{for } n = 0,1,2, \ldots .$$

Thus if the industry real wage growth rate i_L is sufficiently above the rate of growth in industry real output prices i_Y, then time series depreciation will be greater than cross sectional depreciation. Conversely, if $1+i_L < (1+i_Y)^{3/2}$, then time series depreciation will be less than the corresponding cross sectional depreciation for each age of the asset.

The Leontief and Cobb Douglas models serve to illustrate how the plain vanilla Solow-Harper vintage production function model works. If industry wages increase more rapidly than industry output prices, then there is a general tendency for the useful life of each vintage asset to contract (or at least not increase) and a general tendency for real time series depreciation to exceed the corresponding cross sectional depreciation. These two tendencies mean that deterioration rates that are estimated using cross sectional data at any point in time may underestimate the corresponding (more important) time series depreciation rates in an environment where input prices are increasing more rapidly than output prices.

In the following sections, we will generalize this basic Solow-Harper machines model in various directions.

7. Increasing Maintenance Requirements and the Asset Retirement Decision

As a machine or structure ages, it will generally require increasing maintenance and materials inputs to keep the asset functioning properly.[34] This aspect of capital can be captured in the Solow-Harper machines model by assuming that the vintage production functions f_n have the following structure:

(48) $f_n(L,K) \equiv f[L/(1+\gamma)^n, (1-\delta)^n K]$; $n = 0,1,2, \ldots$

where $\gamma > 0$ is the rate of increase in labor and materials that is required to properly maintain a unit of capital (compared to a unit that is one period younger) and as usual, $0 \leq \delta < 1$ is the one period (geometric) deterioration rate. Note that the vintage production function f that applies to newly installed capital is equal to f_0 so that $f_0(L, K)$ equals $f(L,K)$. Thus as capital ages in this model, it deteriorates at the rate δ and it also requires increasing inputs of input L in order to attain the output levels of previous periods.

We now calculate the unit profit function $\pi_n(p,w)$ for a firm in this industry that is using one unit of capital that was installed n periods ago:

(49) $\pi_n(p,w) \equiv \max_L \{pf_n(L, 1) - wL\}$ $n = 0,1,2, \ldots$
$= \max_L \{pf[L/(1+\gamma)^n, (1-\delta)^n 1] - wL\}$ using (48)
$= \max_L \{p(1-\delta)^n f[L/(1-\delta)^n(1+\gamma)^n, 1] - wL(1-\delta)^n/(1-\delta)^n\}$
using the linear homogeneity of f
$= (1-\delta)^n \max_L \{pf[L/(1-\delta)^n(1+\gamma)^n, 1] - wL(1+\gamma)^n /(1-\delta)^n(1+\gamma)^n\}$
$= (1-\delta)^n \max_{L^*} \{pf[L^*, 1] - w(1+\gamma)^n L^*\}$ letting $L^* \equiv L/(1-\delta)^n(1+\gamma)^n$
$= (1-\delta)^n \pi(p,(1+\gamma)^n w)$ using definition (2).

Thus the amount of gross profits earned today by a unit of capital installed n periods ago, $\pi_n(p,w)$, is equal to $(1-\delta)^n$ times $\pi(p,(1+\gamma)^n w)$, which is the amount earned by a newly installed unit of capital except that the current industry wage rate is adjusted upwards from w to $(1+\gamma)^n w$. Making assumptions (28) about future expected prices and using (5) and (49) leads to the following expressions for the asset price of a newly installed unit of capital:

(50) $P_0^0 = \sum_{n=0}^{\infty} (1-\delta)^n \pi[(1+i_Y)^n p^0, (1+\gamma)^n(1+i_L)^n w^0]/(1+r)^n$.

Comparing (50) with the corresponding expression (5) for our earlier model, which is a special case of the present model with γ equal to zero, we see that the increasing maintenance requirements due to asset ageing have the effect of *augmenting the rate of growth in real wages*; i.e., instead of the anticipated real price of labor n periods into the future being $(1+i_L)^n w^0$, it is the larger price $(1+\gamma)^n(1+i_L)^n w^0$. This means that the increasing maintenance model makes it more likely that (one plus) the rate of growth of augmented wages, $(1+\gamma)(1+i_L)$, will be larger than the growth rate for industry output prices, $(1+i_Y)$, thus reinforcing the general tendency for the useful life of each vintage

[34] See Solow, Tobin, Weizsäcker and Yaari (1966; 112).

asset to contract over time and the general tendency for time series depreciation to exceed the corresponding cross sectional depreciation.

8. Capital Augmenting Technical Change

In this section, we determine how the capital augmentation model of technical progress works in the machine model context. Thus suppose at the beginning of period 0, a new capital asset appears which is identical to the old one but its ability to provide capital services in each period increases by the factor $\alpha_K > 1$. Let P_0^{0*} denote the price of this new improved model. Then at the beginning of period 0, one unit of a new improved machine would be expected to yield the following discounted stream of future rentals:

$$(51)\; P_0^{0*} \equiv \sum_{n=0}^{\infty} \pi_n[p^n, w^n]\alpha_K/(1+r)^n$$
$$= \alpha_K\, P_0^{0}$$

where P_0^0 is the price of an old model purchased at the beginning of period 0. Thus in this simple form of capital augmenting technical progress, the ratio of the stock price of the improved model to the stock price of the unimproved model, P_0^{0*}/P_0^0, is equal to α_K, *the capital augmentation factor*. Hence, to make units of the new improved model comparable to the old models, we need only divide its price by α_K and multiply the number of units in use by α_K and equations (5)-(23) can still be applied to the relabeled new model and the vintage old models. Hence if the new model is appropriately quality adjusted by the national statistical agency, then the simple machines model outlined in section 2 above can be applied with no complications in this capital augmentation model.

However, there is an empirical problem with this capital augmentation model that is shared by our original machines model. The problem has been noted above: in the normal case where the wage rate is increasing more rapidly than the industry output price, then industry profits will decrease over time and in most cases, as time marches on, the expected life of *all* machines in use will steadily decline if expectations about future prices are realized. The assumption of capital augmenting technical progress does not change this feature of the model as we have just shown.[35] However, Harper[36] showed us the way out of this problem: simply assume that a machine innovation is not only capital augmenting but that it is also labor augmenting.[37]

9. More General Augmentation Models

Before the algebra of a more general augmentation model is developed, an example may help clarify matters. Consider a truck innovation. The new improved truck is bigger and more fuel efficient. The bigger size means that the number of ton miles that can be

[35] This problem of a steadily declining expected life of a new asset persists in the more realistic machine model of the previous section where there is a rising cost of maintenance over time.
[36] "It is clear that two machines are not homogeneous if one of them embodies an improvement that enhances labor productivity." Michael J. Harper (2007).
[37] Solow, Tobin, Weizsäcker and Yaari (1966; 82) also consider more general types of labor and capital augmenting technical progress in their Leontief model.

transported in a given period of time is, say, α_K times bigger than the best older vintage truck model. This is a capital augmentation effect. However, the new truck still requires only one driver and so the amount of output that can be hauled in a period per unit of labor increases by α_L (a number greater than one). This is a labor augmentation effect.[38]

We now consider how to model the effects of capital and labor augmenting technical change in our simple production function model. Recall that the preinnovation production function was $Y = f(L,K)$ and the corresponding unit profit function was $\pi(p,w)$ defined by (2). The production function for the new model is $Y = f(\alpha_L L, \alpha_K K)$ where $\alpha_L > 1$ and $\alpha_K \geq 1$ are the labor and capital augmentation factors respectively.[39] The new gross profit or operating surplus for a firm in the industry that uses 1 unit of the new model is now defined as

$$
\begin{aligned}
(52)\ \pi^*(p,w) &\equiv \max_L \{pf(\alpha_L L,\alpha_K) - wL\} \\
&= \max_L \{pf(\alpha_K\alpha_K^{-1}\alpha_L L, \alpha_K) - w(\alpha_L)^{-1}\alpha_L L\} \\
&= \max_L \{p\alpha_K f(\alpha_K^{-1}\alpha_L L, 1) - w(\alpha_L)^{-1}\alpha_L L\} \\
&\qquad \text{using the linear homogeneity of } f \\
&= \alpha_K \max_L \{pf(\alpha_K^{-1}\alpha_L L, 1) - w(\alpha_L)^{-1}\alpha_K^{-1}\alpha_L L\} \\
&= \alpha_K \max_{L^*} \{pf(L^*, 1) - w(\alpha_L)^{-1}L^*\} \qquad \text{where } L^* \equiv \alpha_L\alpha_K^{-1}L \\
&= \pi[p, (\alpha_L)^{-1}w]\,\alpha_K
\end{aligned}
$$

where $\pi(p,w)$ is the old preinnovation unit profit function. Thus the effect of the labor augmenting technical change is to lower the effective wage rate for a new model from w to w/α_L and then the effect of the capital augmenting technical change is to blow up the labor augmentation factor adjusted profits $\pi[p, (\alpha_L)^{-1}w]$ by the factor α_K. The value of the new improved machine at the beginning of period 0, P_0^{0**}, will be the expected value of the discounted stream of future profits:

$$
\begin{aligned}
(53)\ P_0^{0**} &= \textstyle\sum_{n=0}^{\infty} \pi[p^0(1+i_Y)^n,(\alpha_L)^{-1}w^0(1+i_L)^n]\alpha_K /(1+r)^n \\
&> \textstyle\sum_{n=0}^{\infty} \pi[p^0(1+i_Y)^n,w^0(1+i_L)^n] /(1+r)^n \\
&\qquad \text{using } \alpha_L > 1,\ \alpha_K \geq 1 \text{ and the monotonicity properties of } \pi \\
&= P_0^0
\end{aligned}
$$

where P_0^0 is the value of a preinnovation model installed at the beginning of period 0; see (5) above with future period wages and prices defined by (28).[40] Note that (53) indicates

[38] If the fuel economy of the new truck were also improved, there would be a materials augmentation effect as well and if the maintenance requirements of the new model truck were diminished, there would be a maintenance expenditures augmentation effect as well. All of these augmentation effects are regarded as labor augmenting effects in our highly aggregated model.

[39] If $\alpha_L = \alpha_K$, then we have Hicks neutral technical change. In the context of a constant returns to scale production function, this type of technical change is equivalent to output augmenting technical change.

[40] Thus the new model and the old model can coexist at the beginning of period 0. However, if the augmentation factors for the new model, α_L and α_K, are sufficiently large, then what is likely to happen is that the industry output price will fall. If the fall is sufficiently large, then the discounted stream of profits generated by the preinnovation model, P_0^0, will fall below the cost of producing this model and no new preinnovation models will be sold. However, the older vintages of the preinnovation model can continue to

that the effect of a large labor augmentation factor α_L will be to scale down the future period anticipated wage rates, $w^0(1+i_L)^n$, and hence for most technologies (and the Leontief technology in particular), *this will in turn increase the expected life of the new model compared to the old model*. Thus for Leontief technologies, (34) and (35) defined the period T where the old asset would be retired; i.e., T was such that $\pi[p^0(1+i_Y)^T,w^0(1+i_L)^T]$ was approximately equal to zero. For the new post innovation model, the new age of retirement is T^* where T^* is such that $\pi[p^0(1+i_Y)^{T*},(\alpha_L)^{-1}w^0(1+i_L)^{T*}]$ is approximately equal to zero; i.e., T^* is such that

$$(54)\ \pi[p^0(1+i_Y)^{T*},(\alpha_L)^{-1}w^0(1+i_L)^{T*}] \approx 0.$$

Now the importance of labor saving technical progress can be seen: *it has the effect of increasing the expected useful life of a new model compared to the previous model*; i.e., since α_L is greater than 1 and $\pi(p,w)$ is decreasing in w, we see that T^* defined by (54) will generally be greater than the T defined by (55):

$$(55)\ \pi[p^0(1+i_Y)^T,w^0(1+i_L)^T] \approx 0.$$

In the following section, we indicate some possible methods for aggregating capital services when a new model appears.

10. The Aggregation of Capital Services when New Models Appear

Suppose a new model appears on the market in period 1 where the new model is both labor and capital augmenting[41] compared to the existing model that is being used by an industry. How can a capital services aggregate be calculated under these conditions? The aggregation problem is much more severe than the aggregation problem when we just had capital augmenting technical change because in the latter case, we had comparable units of measurement; i.e., the new and the old model differed in only one characteristic, which in principle, could be measured. Now we have old and new models that differ in two significant characteristics; i.e., we have a *new goods problem* rather than a simple scaling of quantities problem.

How could this new goods problem be addressed? To explain the problem in a highly simplified manner, consider the problem of forming an industry capital services aggregate going from period 0 to 1. Suppose $q_1^t > 0$ units of the preinnovation model were purchased by the industry at the beginning of period t for t = 0,1 and the corresponding rental (or user benefit) prices for this new unit were $\pi(p^0,w^0) \equiv u_1^0 > 0$ in period 0 and $\pi(p^1,w^1) \equiv u_1^1 > 0$ in period 1. A new model is introduced to the industry in period 1 and $q_2^1 > 0$ units of it were purchased at the beginning of period 1 at the user benefit price $\pi(p^1,\alpha_L^{-1}w^1)\alpha_K \equiv u_2^1 > 0$. The quantity purchased of the new model in period 0 is obviously $q_2^0 \equiv 0$ but the problem is: what should we take as the price of the

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generate rents for a while until the point in time when cash flow becomes negative. Note that this point is never reached for a Cobb Douglas technology.

[41] If the new model is just capital augmenting, there are no new difficulties; see section 8 above.

new model, u_2^0, in period 0? In this situation, it seems reasonable to apply the reservation price methodology developed by Hicks (1940; 114) in the consumer context to the present industrial situation. A reasonable user benefit price for the new model in period 0 is simply $\pi(p^0,\alpha_L^{-1}w^0)\alpha_K \equiv u_2^0 > 0$ since at this price, industrial demanders for the new type of capital would just be indifferent to purchasing the services of the new model if they faced this period 0 rental price for the new model in period 0. With these definitions, the Fisher (1922) price and quantity indexes[42], P_F and Q_F, can be used to form indexes of price and quantity change going from period 0 to 1 for a capital services aggregate of new and old models:[43]

(56) $P_F(u^0,u^1,q^0,q^1) \equiv [u^1\cdot q^0\, u^1\cdot q^1/u^0\cdot q^0\, u^0\cdot q^1]^{1/2}$;
(57) $Q_F(u^0,u^1,q^0,q^1) \equiv [u^0\cdot q^1\, u^1\cdot q^1/u^0\cdot q^0\, u^1\cdot q^0]^{1/2}$

where $u^t \equiv [u_1^t,u_2^t]$ and $q^t \equiv [q_1^t,q_2^t]$ for $t = 0,1$.[44] A similar methodology will work for forming a capital stock aggregate. Of course, the practical problems associated with implementing the above suggested approach[45] are formidable but at least a general theoretical approach has been suggested.

An example of the above procedure for forming a capital services aggregate may help to see whether it is reasonable or not. Suppose that the unit profit function for the preinnovation technology has the following Leontief functional form:

(58) $\pi(p,w) \equiv 4p - w$.

Thus one unit of newly purchased capital can produce 4 units of output using 1 unit of labor. Suppose further that the industry output prices in period 0 and 1 are $p^0 \equiv 1$ and $p^1 \equiv 1$ respectively and the industry variable input price or wage rates in period 0 and 1 are $w^0 \equiv 1$ and $w^1 \equiv 1$ respectively. The capital service price for a new unit of preinnovation capital is $\pi(p^0,w^0) \equiv 4p^0 - w^0 = 4 - 1 = 3 \equiv u_1^0$ in period 0 and is $\pi(p^1,w^1) \equiv 4p^1 - w^1 = 4 - 1 = 3 \equiv u_1^1$ in period 1. Suppose further that the industry purchases one unit of preinnovation capital in each period so that $q_1^0 \equiv 1$ and $q_1^1 \equiv 1$.

---

[42] We choose the Fisher price and quantity index as our preferred index number formula because it can be justified from many alternative approaches to index number theory; see Chapters 15-19 in the ILO (2004).

[43] For simplicity, we have omitted aggregating over the vintages of the old model. For a general approach to aggregating over vintages, see Diewert and Lawrence (2000). The above methodology for dealing with the new goods problem in the producer context was suggested by Fisher and Shell (1972; 101) and Diewert (1980; 498-499).

[44] For simplicity, we have not explicitly accounted for older assets in this simplified presentation. However, using the material on Hicks' aggregation presented in section 2, it can be seen that q_1^1 can be interpreted as the aggregate (over all vintages) quantity of the preinnovation type of capital.

[45] The main problem is the estimation of the user benefits, the u_n^t, and in particular, the estimation of the imputed period 0 benefit for the new model, $\pi(p^0,\alpha_L^{-1}w^0)\alpha_K \equiv u_1^0$. Another difficult problem is the determination of T and T^*, the expected length of life for the preinnovation model and the new model respectively. However, practical statisticians are used to making rough and ready approximations to a target index.

In period 1, the new model becomes available. The unit profit function for the new model has the following Leontief functional form:

(59) $\pi(p,\alpha_L^{-1}w)\alpha_K \equiv [4p - w/2]2 = 8p - w$.

Thus the capital augmentation factor for the new model is $\alpha_K \equiv 2$ and the labor augmentation factor for the new model is $\alpha_L \equiv 2$ as well. Thus one unit of new model capital can produce 8 units of output using 1 unit of labor.[46] The capital service user charge for a new unit of new model capital in period 1 is $\pi(p^1,\alpha_L^{-1}w^1)\alpha_K = 8p^1 - w^1 = 8 - 1 = 7 \equiv u_2^1$. Suppose that the industry purchases one unit of the new model capital in period 1 (and no units in period 0) so that $q_2^0 \equiv 0$ and $q_2^1 \equiv 1$. The imputed capital service charge for a unit of new model capital in period 0 is $\pi(p^0,\alpha_L^{-1}w^0)\alpha_K = 8p^0 - w^0 = 8 - 1 = 7 \equiv u_2^0$.

Consolidating the above information, the period 0 and 1 price of capital services vectors are $u^0 \equiv [u_1^0,u_2^0] = [3,7]$ and $u^1 \equiv [u_1^1,u_2^1] = [3,7]$ respectively and the period 0 and 1 quantity of capital services vectors are $q^0 \equiv [q_1^0,q_2^0] = [1,0]$ and $q^1 \equiv [q_1^1,q_2^1] = [1,1]$ respectively. Inserting this information into the Fisher formulae (56) and (57), we find that $P_F(u^0,u^1,q^0,q^1) = 1$ and $Q_F(u^0,u^1,q^0,q^1) = 10/3$. The fact that the Fisher price index is 1 follows from the fact that the period 0 capital services price vector u^0 equals its period 1 counterpart u^1 in this example.[47] We now create period 0 and 1 industry capital services aggregates. The value of industry capital services for time period t is

(60) $V_K^t \equiv u^t \cdot q^t$; $t = 0,1$.

The price and quantity of industry capital services in period $t = 0,1$, P_K^t and Q_K^t, are defined in general as follows:

(61) $P_K^0 \equiv 1$; $P_K^1 \equiv P_F(u^0,u^1,q^0,q^1)$.
(62) $Q_K^0 \equiv V_K^0$; $Q_K^1 \equiv V_K^0\, Q_F(u^0,u^1,q^0,q^1) = V_K^1 / P_F(u^0,u^1,q^0,q^1)$.

For our particular example, we have:

(63) $V_K^0 = 3$; $V_K^1 = 10$; $P_K^0 = 1$; $P_K^1 = 1$; $Q_K^0 = 3$; $Q_K^1 = 10$.

As a rough check on whether the above decomposition of the value of capital services into price and quantity components is reasonable, we could ask what happens if the assumption that α_L is equal to 2 is replaced by the assumption that α_L is equal to 1; i.e., the labor augmentation factor for the new technology is set equal to one, so that the new technology is actually a scalar multiple of the old technology. Working through the details of this new hypothesis, we find that we still have P_K^0 equals 1, P_K^1 equals 1, and Q_K^0 equals 3 but now Q_K^1 equals 9 compared to its previous level of 10. Thus it can be

[46] Minus labour demand is obtained by differentiating the profit function $8p - w$ with respect to w. Since this derivative is -1, the demand for labour when one unit of new capital is used is 1.
[47] Thus the Paasche and Laspeyres price indexes are also 1 for this example.

seen that the effect of having a labor augmentation factor greater than one is to *increase* the quantity of capital services compared to a situation where the labor augmentation factor is equal to one. This seems entirely reasonable.

Instead of using the above Hicksian imputation procedure to determine the price and quantity of industry capital services, it is possible to use a second simpler procedure. In this second strategy, the industry price index for capital services is determined by calculating a Fisher price index for all the types of capital services that are actually in use by the industry for the two periods under consideration. Denote this new Fisher price index by $P_F^*(u^0,u^1,q^0,q^1)$. Using this restricted domain Fisher price index, the price and quantity of industry capital services in period t = 0,1, P_K^t and Q_K^t, are now defined as follows:[48]

(64) $P_K^0 \equiv 1$; $P_K^1 \equiv P_F^*(u^0,u^1,q^0,q^1)$.

(65) $Q_K^0 \equiv V_K^0$; $Q_K^1 \equiv V_K^1 / P_F^*(u^0,u^1,q^0,q^1)$.

In the example above, there is only one type of capital service that is present in both periods (new units of the preinnovation capital) and so $P_F^*(u^0,u^1,q^0,q^1)$ reduces to the single price ratio u_1^1/u_1^0 which is equal to 1. Since the restricted domain Fisher price index $P_F^*(u^0,u^1,q^0,q^1)$ and the full Fisher price index $P_F(u^0,u^1,q^0,q^1)$ using Hicksian imputations are both equal to 1 for this example, our second suggested decomposition procedure gives the same answer for our example as the first procedure.

It is interesting to note that in the above example, there is no Total Factor Productivity Growth in the industry that is using the capital; any TFP growth that occurs in the economy as a result of the innovation must take place in the producing industry rather than the receiving industry.[49] Upon reflection, we see that this constant industry TFP result is plausible, given that: (i) we have not assumed any technical progress for the two firms in our industry; (ii) we have assumed constant returns to scale production functions for each firm and (iii) industry output and input prices are assumed to be the same for each firm in each period[50] so that efficiency reallocation effects across firms are zero in our industry model.

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[48] Diewert (1980; 500-501) suggested this procedure.

[49] To see this for our example, note that the single firm using one unit of preinnovation capital produces 4 units of output and uses 1 unit of labor and 3 units of capital. The period 1 industry consists of a firm that uses 1 unit of preinnovation capital and another firm that uses 1 unit of new model capital. The period 1 total output is 4+8 = 12, total labor input is 1+1 = 2 and total capital input is 3+7 = 10. The Fisher index of output growth between the two periods is just industry output produced in period 1 divided by industry output produced in period 0 or 12/4 = 3. Since the price of labor remains constant and the price of aggregate capital also remains constant going from period 0 to 1, the Fisher index of industry input growth is just equal to the input cost ratio, (2+10)/(1+3) = 12/4 = 3. Thus if TFP Growth is defined as the industry Fisher output quantity index divided by the corresponding Fisher input quantity index, we see that TFPG = 3/3 = 1 and thus there was no improvement in (using) industry TFP going from period 0 to 1 using our suggested quality adjustment procedure.

[50] We are regarding new and old capital services as separate inputs.

The above example had the property that the industry output price and the industry input prices remained constant before and after the new model was introduced. In many cases where the innovation is dramatically cost saving, it is likely that the increased efficiency of the new "wonder" model will cause the price of output to fall in the industry that uses the new model. Thus we now modify the above example to reflect this reality. We leave the period 0 data unchanged and the period 1 wage rate remains fixed at $w^1 = 1 = w^0$ but now the industry output price falls from $p^0 = 1$ to $p^1 = ½$. Using (58), the rent that a new unit of preinnovation capital can earn in period 1 is $\pi(p^1,w^1) \equiv 4p^1 - w^1 = 4(1/2) - 1 = 1 \equiv u_1^1$. As before, we assume that no units of the new model were purchased in period 0 but one unit of the new model is purchased by the industry in period 1 so that $q_2^0 = 0$ and $q_2^1 = 1$. The rent that a new model can earn in period 1 is $\pi(p^1,\alpha_L^{-1}w^1)\alpha_K = 8p^1 - w^1 = 8(1/2) - 1 = 3 \equiv u_2^1$. The imputed capital service user charge for a unit of new model capital in period 0 is $\pi(p^0,\alpha_L^{-1}w^0)\alpha_K = 8p^0 - w^0 = 8 - 1 = 7 \equiv u_2^0$ as in the previous example.

Consolidating the above information, the period 0 and 1 price of capital services vectors are $u^0 \equiv [u_1^0,u_2^0] = [3,7]$ and $u^1 \equiv [u_1^1,u_2^1] = [1,3]$ respectively and the period 0 and 1 quantity of capital services vectors are $q^0 \equiv [q_1^0,q_2^0] = [1,0]$ and $q^1 \equiv [q_1^1,q_2^1] = [1,1]$ respectively. Inserting this information into the Fisher formulae (56) and (57), we find that $P_F(u^0,u^1,q^0,q^1) = .3651$. Using the general formulae defined by (60)-(62), we find that the value, price and quantity of capital services for the two periods is:

(66) $V_K^0 = 3$; $V_K^1 = 4$; $P_K^0 = 1$; $P_K^1 = 0.3651$; $Q_K^0 = 3$; $Q_K^1 = 10.9545$.

Note that in this example, the aggregate price of capital services in period 1, P_K^1, equals 0.3651, which is between the price ratio for the preinnovation model, $u_1^1/u_1^0 = 1/3 = .3333$, and the price ratio for the new model, $u_2^1/u_2^0 = 3/7 = .4286$.

It is of some interest to calculate industry Total Factor Productivity growth for this example. Firm 1 in period 0, using one unit of preinnovation capital produces 4 units of output, uses 1 unit of labor and 3 units of capital and these are the industry totals for period 0. The prices of labor and capital are both 1 in period 0. Firm 1 in period 1, using 1 unit of old model capital, produces 4 units of output, uses one unit of labor and $1/P_K^1$ units of capital services in constant quality units. Firm 2, using 1 unit of new model capital in period 1, produces 8 units of output, uses one unit of labor and $3/P_K^1$ units of capital services in constant quality units. Thus industry output in period 1 is 12, labor input is 2 and aggregate capital services input is $4/P_K^1$, which is equal to 10.9545 and these are the industry totals for period 1. The industry price of labor is 1 and the industry price of aggregate capital services is P_K^1 equal to 0.3651 in period 1. The Fisher index of output growth between the two periods is just industry output produced in period 1 divided by industry output produced in period 0 or 12/4 = 3. The Fisher index of industry aggregate capital and labour growth turns out to equal 3.0452 so that industry TFP growth is equal to 3/3.0452 = 0.9852. Thus in this new example, it appears that the capital using industry experiences a *productivity loss* of 1.5%. However, this loss is due to the small amount of aggregation bias that occurred when we formed the industry capital aggregate. Recall that the period 0 and 1 price of capital services vectors were defined as $u^0 \equiv [u_1^0,u_2^0] = [3,7]$ and $u^1 \equiv [u_1^1,u_2^1] = [1,3]$ respectively and the period 0 and 1 quantity of capital services vectors were defined as $q^0 \equiv [q_1^0,q_2^0] = [1,0]$ and $q^1 \equiv [q_1^1,q_2^1] = [1,1]$ respectively. Now add the price of labour (equal to 1 in both periods) as a third component to u^0 and u^1 and add the quantity of labour (equal to 1 in period 0 and 2 in period 1) to q^0 and q^1 respectively, and compute new Fisher indexes of input. Thus we replace the two stage aggregation of inputs that was used earlier in the paragraph with a single stage procedure. The single stage procedure turns out to equal 3, so that industry TFP growth using the more accurate single stage aggregation is equal to 3/3 = 1.0000 so that again, the industry shows no productivity gain due to the innovation.

Now use the restricted domain method for decomposing the value of capital services into price and quantity components in place of the Hicksian imputation procedure. To implement this second procedure, use the restricted domain Fisher price index $P_F^*(u^0,u^1,q^0,q^1)$, which equals $u_1^1/u_1^0 = 1/3$, in place of the full Fisher price index $P_F(u^0,u^1,q^0,q^1) = 0.3651$. Using $V_K^0 = 3$ and $V_K^1 = 4$ and (64)-(65), we find that

(67) $V_K^0 = 3$; $V_K^1 = 4$; $P_K^0 = 1$; $P_K^1 = 0.3333$; $Q_K^0 = 3$; $Q_K^1 = 12$.

Comparing (67) with (66), we see that the restricted domain procedure has increased the period 1 quantity of capital services Q_K^1 and decreased its price P_K^1. Again industry output in period 1 is 12, labor input is 2 but now capital input is $4/P_K^1 = 12$. The industry price of labor is 1 and the industry price of capital is $P_K^1 = 0.3333$ in period 1. The Fisher index of output growth between the two periods is again industry output produced in period 1 divided by industry output produced in period 0 or 12/4 = 3. The Fisher index of industry input growth turns out to equal 12.9615/4, which is equal to 3.2404 so that industry TFP growth is equal to 3/3.2404 = 0.9258. Thus for this example, the restricted domain method of quality adjustment leads to a 7.4% *drop* in TFP for the capital using industry compared to our earlier estimate of a 1.5% *drop* in TFP using the (two stage) Hicksian imputation method or an estimate of *no change* in TFP using the (single stage) Hicksian imputation method.

Which method of quality adjustment is better? On theoretical grounds, the Hicksian method seems preferable.[51] However, this method will be more difficult for economic statisticians to implement than the restricted domain method and so as a practical matter, we will probably have to be satisfied with this second method.[52]

11. Conclusion

The algebra in the previous sections backs up the points made in the Harper critique of the existing vintage models for forming aggregate capital services. Basically, the existing framework ignores the role of complex forms of technological progress and the fact that

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[51] The second method implicitly relies on a linear aggregation procedure; i.e., value added produced by the new model in period 1 is additively equivalent to value added produced by the older models. The Hicksian imputation procedure using a superlative index number formula like the Fisher formula is consistent with more general forms of economic aggregation; see Diewert (1976).

[52] Example 1 shows that the first and second methods can be close to each other in practice.

capital asset and service prices can be affected by changes in output and input prices. Of particular concern is the implication of Solow, Tobin, von Weizsäcker and Yaari (1966) and Harper's work that the length of life of an asset is not an exogenously given fact but is endogenously determined. This leads to situations where the number of vintages of a particular type of capital in existence varies as time marches on and hence it will be impossible to apply the Diewert Lawrence (2000) methodology where like vintages are compared with like vintages across two time periods using a superlative index number formula. However, the analysis presented in the previous section shows how this situation could be handled: as a vintage disappears, we need only apply the two treatments for the appearance of a new good that were suggested in section 10 in reverse; i.e., we have a disappearing goods problem instead of a new goods problem but the same methodological approaches will work in both cases with some relabelling of the variables.

Another troublesome aspect of the Solow-Harper model was pointed out in section 3 above: in this model, there is no directly observable sequence of used asset prices that can be used in order to determine the deterioration rate δ. The only general way that the economic statistician could determine δ is to either use econometric techniques in order to estimate the parameters of the vintage production functions (or their dual unit profit functions) along with δ. This is a task that national statistical agencies are not well equipped to undertake. Even if the deterioration rate could be estimated econometrically or by engineering studies, the analysis in this paper shows that deterioration is not equal to time series depreciation and it will be extremely difficult to measure depreciation accurately in nonseparable models such as the Solow-Harper model studied in this paper.

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