Sincere Voting in Large Elections

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May 20, 2011

Abstract. Recounting introduces multiple pivotal events in two-candidate elections. In addition to determining which candidate is elected, an individual's vote is pivotal when the vote margin is just at the levels that would trigger a recount. In large elections, the motive to avoid recount cost can become the dominant consideration for rational voters, inducing them to vote sincerely according to their private signals. In environments where elections without recount fail to aggregate information efficiently, a suitably modified election rule with small recount cost can produce asymptotically efficient outcomes with a vanishing small probability of actually invoking a recount.

JEL classification. C72, D72, D82

Keywords. recounting, multiple pivotal events, information efficiency, aggregate uncertainty, conflicting preferences

1. Introduction

More than two centuries ago Condorcet (1875) first articulated the idea that voting groups with diverse information about their alternatives make a better choice the larger the group size. This celebrated Condorcet jury theorem is a statistical proposition based on an early application of the law of large numbers. It is an important result that gives confidence to our belief that large elections can resolve conflicts due to dispersed information and produce good collective decisions.

Although intuitively appealing, the presumed sincere-voting behavior by the electorate has been re-examined by economists who study this topic. Austen-Smith and Banks (1996) first point out that voting according to one's own private signal is generally inconsistent with rationality (see also Feddersen and Pesendorfer, 1996). Since a non-pivotal vote does not affect the outcome and is thus payoff-irrelevant, rational voting behavior requires conditioning one's vote on the information inferred from the vote being pivotal as well as on one's own private information. In a large election, the information inferred from being pivotal can overwhelm one's own private information. Thus, sincere voting generally fails in a large election except for a small fraction of informed voters.[1]

The failure of sincere voting notwithstanding, Feddersen and Pesendorfer (1997) show that in a large two-candidate election the outcome is information-efficient in the sense that almost surely it would remain the same even if all the private information about the candidates became common knowledge. Under any election rule, the outcome in a large election would be determined by the corresponding decisive voter's preference if the private information were perfectly aggregated. For example, under the simple majority rule, the decisive voter has the median preference in the electorate. Similarly, under strategic voting, votes are cast as if the election is close and the decisive voter is indifferent between the two candidates. Even though the fraction of voters whose vote depends on their private signals is small in a large election, their number goes to infinity. It is these voters that determine the election outcome, ensuring that the outcome is information efficient.

Nevertheless, there are environments in which information efficiency fails under

[1] In a two-candidate election model with a continuous payoff state, Feddersen and Pesendorfer (1997) show that the fraction of agents who vote informatively vanishes as the election size becomes arbitrarily large.

strategic voting even though it would obtain had all informed voters voted sincerely. One such environment involves "aggregate uncertainty," where there are partisan voters who randomly split their votes between the two candidates, resulting in uncertainty in realized vote shares even when the number of voters becomes arbitrarily large (Feddersen and Pesendorfer, 1997). Another environment involves conflicting preferences, where the same change in the public belief about a candidate can increase his appeal to some voters but lower his appeal to other voters (Bhattacharya, 2008).

In this paper, we resurrect sincere voting as an equilibrium strategy in large two-candidate elections by introducing other pivotal events in addition to the standard one that determines the eventual winner. Although there are many ways to introduce additional pivotal events, we adopt a model of costly recounting. An election rule in this model is characterized by three thresholds of vote shares for a given candidate and a positive recount cost. If the vote share for the candidate exceeds the largest threshold then that candidate is declared an outright winner; and symmetrically, if the vote share falls below the smallest threshold then the the opposing candidate is declared an outright winner. If the vote share falls between the smallest and the largest thresholds, a "recount" takes place after each voter incurs the recount cost. The candidate is declared the winner if the vote share upon recount is above the middle threshold, and the opposing candidate wins otherwise. We study information aggregation in an environment with two states and conditionally independent private binary signals. To incorporate environments in which information efficiency may not obtain, we introduce both aggregate uncertainty and conflicting preferences into the model. Aggregate uncertainty is modeled by the presence of uninformed voters (who do not receive private signals); the fraction of uninformed voters voting for a given candidate remains random even in large elections. Conflicting preferences is modeled by assuming two kinds of informed voters (who observe private signals). Informed majority voters want to match the selection between the two candidates to the state, while informed minority voters have the opposite preferences. Furthermore, majority voters differ among themselves in their preference intensity in that they require different minimum belief about the state to vote for the matching candidate. Likewise, minority voters also differ in their preference intensity. Our model environment is not a common value election.

We establish sufficient conditions on the election environment such that there exist appropriately chosen thresholds to induce sincere voting (voting according to one's private signal) by all informed voters when the election becomes arbitrarily large. In our model, corresponding to the middle threshold is the standard pivotal event that votes for the two candidates are tied. Costly recounting creates two additional pivotal events: corresponding to the largest threshold is the pivotal event when one more vote for the given candidate would make him an outright winner and one more vote for the opponent would trigger a costly recounting but would not change the winner, and corresponding to the smallest threshold is the symmetric pivotal event. Although the probabilities of the three pivotal events conditional on the state all vanish in the limit, one of them becomes dominant because its conditional probability goes to zero at the slowest rate. This is a consequence of the theory of large deviations, which studies the limit behavior of rare events.[2] Through appropriate choice of the thresholds, we can ensure that conditional on each state the dominant pivotal event is when the matching candidate wins outright versus his winning after a costly recount. Unlike a standard election rule, with recounting there are two separate dominant pivotal events for the two states and we can simultaneously ensure that the two corresponding conditional probabilities go to zero at the same rate. Since all informed voters have the same incentive to avoid the recount cost, regardless of whether they have the same or opposite preferences and regardless of their preference intensities, there is an equilibrium in which all informed voters cast their votes sincerely.

The result that by appropriately introducing additional pivotal events we can induce all informed voters to vote according to their signals is not just a rationale for this intuitively appealing voting behavior in large elections.[3] It also has an important welfare implication in environments where aggregate uncertainty or conflicting preferences prevent any standard election rule from achieving information efficiency. Our election rule with recounting has a better chance of achieving information efficiency than a standard election rule because sincere voting by all informed voters maximizes the extent of information aggregation. Furthermore, we show that the probability of recounting and thus incurring the cost in equilibrium is negligible in large elections.

To use recounting to induce sincere voting by all informed voters, we require

[2]See, for example, Dembo and Zeitouni (1998) for a textbook treatment.

[3]There is some evidence that voters do vote sincerely. See Degan and Merlo (2007).

the common prior belief about the state to be not too extreme and the recounting thresholds to be precisely calibrated. To support sincere voting in equilibrium, the informed voter must believe that the relatively more likely dominant pivotal event is the candidate supported by his signal winning outright versus after a costly recount. This may not be feasible if the strength of the binary signals is overwhelmed by the effect of extreme priors. The requirement to have precisely calibrated recounting thresholds comes from the need to ensure that the probabilities of the relevant pivotal events go to zero at the same rate. It turns out that neither of these requirements is necessary for achieving information efficiency. We show that an equilibrium that approaches information efficiency in large elections exists as long as the recounting thresholds fall within two exogenous bounds that depend only on the extent of the aggregate uncertainty and the precision of the private information. In such equilibrium, voting is "semi-sincere" in the sense that all informed voters vote according to their private signals for one signal realization but only a fraction of the informed voters vote according to their private signals for the other signal realization. Compared to a standard election rule where having a single pivotal event implies that neither of the two signals can be voted sincerely by all informed voters, our election rules with recounting allow more information to be aggregated. As a result, whenever sincere voting would achieve information efficiency, there are election rules with recounting that do the same, even though in equilibrium not all informed voters vote sincerely.

2. A Model of Elections with Recounting

We study an election with a large number $n+1$ of voters to choose between two candidates: $\mathcal{R}$ and $\mathcal{L}$. Denote the share of votes for $\mathcal{R}$ as V. An "election rule" consists of three thresholds $v_{\mathcal{L}}$, $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$, satisfying $v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}}$, and specifies:

1. candidate $\mathcal{R}$ is elected if $V > v_{\mathcal{R}}$;
2. candidate $\mathcal{L}$ is elected if $V < v_{\mathcal{L}}$;
3. a "recount" is triggered at an additive payoff loss of $\delta > 0$ to each voter if $V \in [v_{\mathcal{L}}, v_{\mathcal{R}}]$; and after the recount, candidate $\mathcal{R}$ is elected if $V \geq v_{\mathcal{C}}$ and candidate $\mathcal{L}$ is elected otherwise.

Note that a standard election rule without recounting can be represented as a special case of election rules defined above, with $\delta = 0$. We assume that there is no error in the initial vote count stage or in the the recount stage. Therefore the vote share for $\mathcal{R}$ in the recount stage will be exactly the same as that recorded in the initial count. We do not consider unanimity rules; both $v_{\mathcal{R}}$ and $v_{\mathcal{L}}$ are assumed to be strictly between 0 and 1.

Voters are independently drawn from a large population of potential voters. A fraction $1-\alpha \in (0,1]$ of potential voters are informed voters; the rest are uninformed. Each informed voter observes a conditionally independent binary signal $s \in \{r, l\}$ about the binary state $S \in \{R, L\}$, with

$$\Pr[r|R] = q_r, \quad \Pr[l|L] = q_l, \quad \text{and} \quad q_r > 1 - q_l.$$

The common prior belief of state R among the informed voters is $\mu \in (0,1)$. The assumption that $q_r > 1 - q_l$ implies that an informed voter who observes signal r would revise his belief of state R upward, while one who observes signal l would revise his belief downward. Uninformed voters are introduced to preserve uncertainty about the realized vote share given state S even in large elections. They are non-strategic; a fraction θ of them vote for candidate $\mathcal{R}$ and the remaining fraction $1-\theta$ vote for $\mathcal{L}$.[4] The fraction θ is a random variable distributed on $[\underline{\theta}, \overline{\theta}] \subseteq [0,1]$, with a continuous and positive density function f and corresponding distribution function F. The aggregate uncertainty state θ is independent of the payoff state S.

An informed voter is either a majority voter, with probability $1-\beta \in (0,1]$, or a minority voter, with the complementary probability β. Majority voters want to match the candidate with the state while minority voters have the opposite preferences. Majority voters differ among themselves in preference intensities. Each majority voter is characterized by a parameter t, with the utility from electing candidate $\mathcal{R}$ in state $S = R$ being $1-t$, and the utility from electing candidate $\mathcal{L}$ in state $S = L$ being t. The utility of choosing $\mathcal{R}$ in state L or $\mathcal{L}$ in state R is 0. Hence majority voters with higher values of t have greater preference for candidate $\mathcal{L}$. Each t is independently drawn from a fixed distribution H. We assume that H has a positive density h on the support $[0,1]$.[5] Similarly, each minority voter with preference intensity $\tilde{t}$ gets a utility $-(1-\tilde{t})$ from electing candidate $\mathcal{R}$ in state R, utility $-\tilde{t}$ from electing candidate $\mathcal{L}$ in state L, and 0 from choosing $\mathcal{R}$ in L or $\mathcal{L}$ in R. We assume that each $\tilde{t}$ is drawn from

[4]The uninformed voters are partisan in the sense that they have preferences between the two candidates that cannot be swayed by any evidence. Otherwise, they may optimally choose to abstain from voting. See Feddersen and Pesendorfer (1996).

[5]When t is allowed to lie outside of $[0,1]$, informed voters have a preference between $\mathcal{L}$ and $\mathcal{R}$ independent of the state. The sincere-voting equilibrium that we construct remains valid in this case.

some $\tilde{H}$ with support $[0, 1]$ and a positive density $\tilde{h}$. Minority voters with higher values of $\tilde{t}$ have greater preference for candidate $\mathcal{R}$.[6]

2.1. Strategy and equilibrium

For a given n, our voting game Γ^n is described by the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ and δ, the preference distribution parameters α and β, and the distribution functions F, H and $\tilde{H}$, which are all common knowledge. Ultimately we are interested voting games with large n, so we will ignore all integer problems. The solution concept is Bayesian Nash equilibrium.

Fix some informed voter. Denote as v the number of votes for $\mathcal{R}$ from all voters other than this voter, divided by the total number of votes n from these voters. Let $g^n(\cdot|R)$ and $g^n(\cdot|L)$ represent the density function of v under state R and state L, respectively. The functions $g^n(\cdot|R)$ and $g^n(\cdot|L)$ are derived from the strategies adopted by all other voters. Since no voter observes the identity of the other n voters, and the payoff state S is independent of the uncertainty state θ, the two density functions depend neither on the preference type t or $\tilde{t}$ nor on the private signal s observed by the informed voter. Further, the presence of uninformed voters guarantees that each $g^n(\cdot|\cdot)$ is a strictly positive function.

First, suppose that the informed voter belongs to the majority. Upon observing the private signal $s = r$, his private belief that the state is R becomes

$$P_r = \frac{\mu q_r}{\mu q_r + (1-\mu)(1-q_l)}.$$

There are three events in which his vote is pivotal:

1. $v = v_{\mathcal{L}}$: Voting $\mathcal{R}$ instead of $\mathcal{L}$ triggers a recount, incurring a cost of δ.
2. $v = v_{\mathcal{C}}$: Voting $\mathcal{R}$ instead of $\mathcal{L}$ tilts the election outcome (after the recount) to $\mathcal{R}$. If the state is R, the gain is $1-t$; if the state is L, the loss is t.
3. $v = v_{\mathcal{R}}$: Voting $\mathcal{R}$ instead of $\mathcal{L}$ determines the outcome of the election immediately, saving the recount cost δ.

[6]These specific payoffs are introduced purely to reduce the notation. Our results hold generally when, for the majority voters, the payoff difference between choosing candidate $\mathcal{R}$ and candidate $\mathcal{L}$ in state R is positive and and decreasing in t, while the same payoff difference in state L is negative and decreasing in t. For the minority voters, the payoff difference between candidate $\mathcal{R}$ and candidate $\mathcal{L}$ is negative and increasing in $\tilde{t}$ in state R, and is positive and increasing in $\tilde{t}$ in state L.

Therefore, voting for $\mathcal{R}$ is preferred to voting for $\mathcal{L}$ if

$$\begin{aligned}&P_r[g^n(v_{\mathcal{R}}|R)\delta + g^n(v_{\mathcal{C}}|R)(1-t) - g^n(v_{\mathcal{L}}|R)\delta] \\ &\quad \geq (1-P_r)[g^n(v_{\mathcal{L}}|L)\delta + g^n(v_{\mathcal{C}}|L)t - g^n(v_{\mathcal{R}}|L)\delta]. \end{aligned} \tag{1}$$

If a majority voter with preference intensity t and signal r finds it weakly optimal to vote for candidate $\mathcal{R}$, then voting for $\mathcal{R}$ is strictly optimal for any majority voter with $t' < t$ and the same signal r.

Similarly, let

$$P_l = \frac{\mu(1-q_r)}{\mu(1-q_r) + (1-\mu)q_l}$$

be the posterior belief of the voter that the state is R upon observing signal $s = l$. In this case, voting for $\mathcal{L}$ is preferred to voting for $\mathcal{R}$ if

$$\begin{aligned}&(1-P_l)[g^n(v_{\mathcal{L}}|L)\delta + g^n(v_{\mathcal{C}}|L)t - g^n(v_{\mathcal{R}}|L)\delta] \\ &\quad \geq P_l[g^n(v_{\mathcal{R}}|R)\delta + g^n(v_{\mathcal{C}}|R)(1-t) - g^n(v_{\mathcal{L}}|R)\delta]. \end{aligned} \tag{2}$$

If a majority voter with preference intensity t and signal l finds it weakly optimal to vote for candidate $\mathcal{L}$, voting for $\mathcal{L}$ is strictly optimal for any majority voter with $t' > t$ and the same signal l.

Conditions (**??**) and (**??**) imply that in any Bayesian Nash equilibrium of the game Γ^n, the voting strategy of majority informed voters can be represented by a pair of preference cutoffs $k^n \equiv (k^n_r, k^n_l) \in [0,1]^2$, such that a majority voter with private signal s votes for candidate $\mathcal{R}$ if and only if $t \leq k^n_s$ $(s = r, l)$. In the equilibria we construct below, the expressions in the brackets in (**??**) and (**??**) are both positive. Then, since $P_r > P_l$, we have $k^n_r \geq k^n_l$. Majority voters with preference t such that $k^n_l < t \leq k^n_r$ vote sincerely, i.e., according to their signals. Majority voters with $t > k^n_r$ always vote for candidate $\mathcal{L}$ and those with $t \leq k^n_l$ always vote for $\mathcal{R}$.[7]

Incentive conditions for a minority voter with preference intensity $\tilde{t}$ are identical to (**??**) and (**??**), except that $-(1-\tilde{t})$ replaces $(1-t)$ and $-\tilde{t}$ replaces t. More precisely, in any Bayesian Nash equilibrium of the game Γ^n, the voting strategy of minority informed voters can be represented by a pair of preference cutoffs

[7]The specification of the actions taken by the threshold types is irrelevant to our equilibrium construction. When thresholds are interior these actions can be chosen arbitrarily, but for corner thresholds there might only be a uniquely optimal action for threshold types.

$\tilde{k}^n \equiv (\tilde{k}_r^n, \tilde{k}_l^n) \in [0,1]^2$, such that a minority voter with signal s votes for $\mathcal{R}$ if and only if his preference intensity $\tilde{t} \geq \tilde{k}_s^n$ ($s = r, l$). The cutoff $\tilde{k}_r^n$ is the smallest number in $[0,1]$ that satisfies

$$\begin{aligned} &P_r[g^n(v_{\mathcal{R}}|R)\delta - g^n(v_{\mathcal{C}}|R)(1-\tilde{k}_r^n) - g^n(v_{\mathcal{L}}|R)\delta] \\ &\quad \geq (1-P_r)[g^n(v_{\mathcal{L}}|L)\delta - g^n(v_{\mathcal{C}}|L)\tilde{k}_r^n - g^n(v_{\mathcal{R}}|L)\delta], \end{aligned} \tag{3}$$

or $\tilde{k}_r^n = 1$ if (**??**) cannot be satisfied. The cutoff $\tilde{k}_l^n$ is the largest number in $[0,1]$ that satisfies

$$\begin{aligned} &(1-P_l)[g^n(v_{\mathcal{L}}|L)\delta - g^n(v_{\mathcal{C}}|L)\tilde{k}_l^n - g^n(v_{\mathcal{R}}|L)\delta] \\ &\quad \geq P_l[g^n(v_{\mathcal{R}}|R)\delta - g^n(v_{\mathcal{C}}|R)(1-\tilde{k}_l^n) - g^n(v_{\mathcal{L}}|R)\delta], \end{aligned} \tag{4}$$

or $\tilde{k}_l^n = 0$ if (**??**) is never satisfied.

A strategy profile of this game is represented by $\kappa^n \equiv (k^n, \tilde{k}^n)$. We denote the strategy profile in which all informed voters are voting sincerely by $\kappa_T \equiv ((1,0),(0,1))$. For any strategy profile κ^n, let $z(R,\theta;\kappa^n)$ be the probability that a randomly drawn voter voting for candidate $\mathcal{R}$ in the payoff state R and aggregate uncertainty state θ. This is given by

$$\begin{aligned} z(R,\theta;\kappa^n) = &(1-\alpha)(1-\beta)[q_r H(k_r^n) + (1-q_r)H(k_l^n)] \\ &+ (1-\alpha)\beta[q_r(1-\tilde{H}(\tilde{k}_r^n)) + (1-q_r)(1-\tilde{H}(\tilde{k}_l^n))] + \alpha\theta. \end{aligned} \tag{5}$$

Similarly, the probability that a randomly drawn voter voting for candidate $\mathcal{R}$ in the payoff state L and aggregate uncertainty state θ is

$$\begin{aligned} z(L,\theta;\kappa^n) = &(1-\alpha)(1-\beta)[(1-q_l)H(k_r^n) + q_l H(k_l^n)] \\ &+ (1-\alpha)\beta[(1-q_l)(1-\tilde{H}(\tilde{k}_r^n)) + q_l(1-\tilde{H}(\tilde{k}_l^n))] + \alpha\theta. \end{aligned} \tag{6}$$

We will refer to $z(S,\theta;\kappa^n)$ as the vote share for candidate $\mathcal{R}$ in state S given the strategy profile κ^n.

Given $z(R,\theta;\kappa^n)$ and $z(L,\theta;\kappa^n)$, from the perspective of each individual informed voter, the probability of a vote share v for candidate $\mathcal{R}$ conditional on the payoff state S and the aggregate uncertainty state θ is then given by

$$g^n(v|S,\theta) = \binom{n}{nv} z(S,\theta;\kappa^n)^{nv}(1-z(S,\theta;\kappa^n))^{n(1-v)}, \tag{7}$$

and thus

$$g^n(v|S) = \int_{\underline{\theta}}^{\overline{\theta}} g^n(v|S,\theta) f(\theta)\,\mathrm{d}\theta. \tag{8}$$

A Bayesian Nash equilibrium of Γ^n is a pair of strategies, k^n of majority voters and $\tilde{k}^n$ of minority voters, and the corresponding pivotal probabilities $g^n(v|R)$ and $g^n(v|L)$, such that: (i) the strategy k^n satisfies the incentive conditions (**??**) and (**??**) and $\tilde{k}^n$ satisfies (**??**) and (**??**) given $g^n(v|R)$ and $g^n(v|L)$; and (ii) the pivotal probabilities $g^n(v|R)$ and $g^n(v|L)$ are derived from the strategies k^n and $\tilde{k}^n$ through (**??**), (**??**), (**??**) and (**??**). Existence of an equilibrium can be established with a standard fixed point argument; we skip this step because our results are all constructive.

2.2. Ranking of pivotal events

Fix a strategy profile κ^n and the implied vote share functions $z(S,\theta;\kappa^n)$. In a large election, the probability that the actual vote share equals a particular value v is vanishingly small. A key observation of this paper is that the *rates* at which the probabilities of different pivotal events go to zero are different, so that in the limit some pivotal events are infinitely more likely to occur than others. Calculating the rate of convergence is therefore an important part of the analysis of large elections with multiple pivotal events.

If voters knew the aggregate uncertainty state θ, then the probability that the vote share equals v is given by the binomial probability $g^n(v|S,\theta)$ in equation (**??**). Using Stirling's approximation formula for the binomial coefficient, we have

$$g^n(v|S,\theta) = \frac{\phi_v^n}{\sqrt{2\pi v(1-v)n}} I\left(v; z(S,\theta;\kappa^n)\right)^n, \tag{9}$$

where

$$I(v;z) = \left(\frac{z}{v}\right)^v \left(\frac{1-z}{1-v}\right)^{1-v},$$

and $\lim_{n\to\infty} \phi_v^n = 1$. The function $-\log I(v;z)$ is known as the "rate function" or "entropy function" in the theory of large deviations. It determines the rate at which the probability $g^n(v|S,\theta)$ goes to zero. In particular, if there are two events v and v' such that $I(v;z(S,\theta;\kappa^n)) > I(v';z(S,\theta;\kappa^n))$, then

$$\lim_{n\to\infty} \frac{g^n(v'|S,\theta)}{g^n(v|S,\theta)} = \left(\frac{I(v';z(S,\theta;\kappa^n))}{I(v;z(S,\theta;\kappa^n))}\right)^n = 0.$$

In our model an informed voter does not know the aggregate uncertainty state θ. The probability of a certain pivotal event v in the two states $g^n(v|S)$ is the integral of $g^n(v|S,\theta)$ over all possible aggregate uncertainty states. The following lemma shows that in determining the rate of convergence, only the θ which maximizes the function $I(v;z(S,\theta;\kappa^n))$ matters.

Lemma 1. *Let $\theta(v,S) \equiv \arg\max_{\theta\in[\underline{\theta},\overline{\theta}]} I(v;z(S,\theta;\kappa^n))$. For any v, v' and any two payoff states S, S',*

$$\lim_{n\to\infty} \frac{g^n(v|S)}{g^n(v'|S')} = \frac{f(\theta(v,S))}{f(\theta(v',S'))} \lim_{n\to\infty} \frac{g^n(v|S,\theta(v,S))}{g^n(v'|S',\theta(v',S'))}.$$

Proof. The function $I(v;z)$ is increasing in z for $z < v$ and decreasing in z for $z > v$, attaining a maximum at $z = v$. Since $z(S,\theta;\kappa^n)$ is strictly increasing in θ, $I(v;z(S,\theta;\kappa^n)$ is decreasing in θ for $\theta < \theta(v,S)$ and increasing in θ for $\theta > \theta(v,S)$. Let $B_\epsilon(v,S) \subset [\underline{\theta},\overline{\theta}]$ be a small interval of width ϵ that contains $\theta(v,S)$. Specifically, if $\theta(v,S) = \underline{\theta}$, choose $B_\epsilon(v,S) = [\underline{\theta},\overline{b})$ where $\overline{b} = \underline{\theta} + \epsilon$; and if $\theta(v,S) = \overline{\theta}$, choose $B_\epsilon(v,S) = (\underline{b},\overline{\theta}]$ where $\underline{b} = \overline{\theta} - \epsilon$. If $\theta(v,S)$ is interior, choose $B_\epsilon(v,S) = (\underline{b},\overline{b})$ such that $\overline{b} - \underline{b} = \epsilon$ and $I(v;z(S,\underline{b};\kappa^n)) = I(v;z(S,\overline{b};\kappa^n))$. Denote $B^c_\epsilon(v,S) = [\underline{\theta},\overline{\theta}]\backslash B_\epsilon(v,S)$ to be the complement of $B_\epsilon(v,S)$. Note that $I(v;z(S,\theta;\kappa^n)) > I(v;z(S,\theta';\kappa^n))$ for any $\theta \in B_\epsilon(v,S)$ and $\theta' \in B^c_\epsilon(v,S)$.

For any pivotal event v and any state S, we have

$$\int_{B^c_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta < g^n(v|S,\theta'_n)\Pr[\theta \in B^c_\epsilon(v,S)],$$

where θ'_n is equal to $\underline{b}$ or $\overline{b}$.

Continuity of $g^n(v|S,\cdot)$ also implies that there is a unique $\hat{\theta}_n \in B_\epsilon(v,S)$ such that

$$\int_{B_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta = g^n(v|S,\hat{\theta}_n)\Pr[\theta \in B_\epsilon(v,S)].$$

We further claim that $\lim_{n\to\infty}\hat{\theta}_n = \theta(v,S)$. To see this, note that by definition

$$\lim_{n\to\infty}\int_{B_\epsilon(v,S)} \frac{g^n(v|S,\theta)}{g^n(v|S,\hat{\theta}_n)} f(\theta)\,\mathrm{d}\theta = \Pr[\theta \in B_\epsilon(v,S)],$$

which is only possible if $\hat{\theta}_n$ converges to $\theta(v,S)$ because from the fact that $\theta(v,S)$ maximizes $I(v;z(S,\theta;\kappa^n))$, we must have $\lim_{n\to\infty} g^n(v|S,\theta)/g^n(v|S,\theta(v,S)) = 0$ for all $\theta \neq \theta(v,S)$.

From the two conditions above, we obtain that for any ϵ positive,

$$\lim_{n\to\infty} \frac{\int_{B^c_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta} \leq \lim_{n\to\infty} \frac{g^n(v|S,\theta'_n)\Pr[\theta\in B^c_\epsilon(v,S)])}{g^n(v|S,\hat{\theta}_n)\Pr[\theta\in B_\epsilon(v,S)]} = 0, \tag{10}$$

where the equality follows because $\lim_{n\to\infty} g^n(v|S,\theta')/g^n(v|S,\theta) = 0$ whenever $\theta' \in B^c_\epsilon(v,S)$ and $\theta \in B_\epsilon(v,S)$, and because $\hat{\theta}_n$ is bounded away from θ'_n.

For any v, v' and S, S',

$$\lim_{n\to\infty} \frac{g^n(v|S)}{g^n(v'|S')} = \lim_{n\to\infty} \frac{\int_{\underline{\theta}}^{\overline{\theta}} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta}{\int_{\underline{\theta}}^{\overline{\theta}} g^n(v'|S',\theta)f(\theta)\,\mathrm{d}\theta} = \lim_{n\to\infty} \frac{\int_{B_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',S')} g^n(v'|S',\theta)f(\theta)\,\mathrm{d}\theta},$$

where the last equality follows from (**??**). The above holds for any ϵ positive and thus

$$\lim_{n\to\infty} \frac{g^n(v|S)}{g^n(v'|S')} = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{\int_{B_\epsilon(v,S)} g^n(v|S,\theta)f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',S')} g^n(v'|S',\theta)f(\theta)\,\mathrm{d}\theta}.$$

Reversing the limit order and calculating the inner limit using l'Hopital's rule, we obtain:

$$\lim_{n\to\infty} \frac{g^n(v|S)}{g^n(v'|S)} = \lim_{n\to\infty} \frac{g^n(v|S,\theta(v,S))f(\theta(v,S))}{g^n(v'|S,\theta(v',S'))f(\theta(v',S'))}. \qquad \blacksquare$$

Lemma **??** implies that for any pivotal events v, v' and any payoff states S, S', the ratio $g^n(v|S)/g^n(v'|S')$ has a limit point different from zero or infinity if and only if there is a limit point κ of κ^n such that

$$I\left(v; z(S,\theta(v,S);\kappa)\right) = I\left(v'; z(S',\theta(v',S');\kappa)\right). \tag{11}$$

We call this an "equal-rate condition." Furthermore, since $I(v;z)$ is increasing in z for $z < v$ and decreasing in z for $z > v$, and since $z(S,\theta;\kappa)$ is increasing in θ, we have

$$\theta(v,S) = \arg\min_{\theta\in[\underline{\theta},\overline{\theta}]} |z(S,\theta;\kappa) - v|.$$

For example, when $z(S,\overline{\theta};\kappa) < v \leq v' < z(S',\underline{\theta};\kappa)$, then $\theta(v,S) = \overline{\theta}$ and $\theta(v',S') = \underline{\theta}$. In this case, the ratio $g^n(v|S)/g^n(v'|S')$ has a limit point different from zero or infinity if and only if $I(v;z(S,\overline{\theta};\kappa)) = I(v';z(S',\underline{\theta};\kappa))$.

3. Sincere Voting

Sincere voting means $k_r^n = 1$ and $k_l^n = 0$ for majority voters and $\tilde{k}_r^n = 0$ and $\tilde{k}_l^n = 1$ for minority winners. This cannot be an equilibrium under any standard voting rule. To see this, note that by setting $\delta = 0$ in (**??**) and (**??**), the incentive conditions cannot be satisfied for $k_r^n = 1$ and $k_l^n = 0$, even if we allow $g^n(v_{\mathcal{C}}|R)$ and $g^n(v_{\mathcal{C}}|L)$ to take on any values. The same is true for (**??**) and (**??**) for minority voters.

With recounting, we have three pivotal events, $v_{\mathcal{L}}$, $v_{\mathcal{C}}$, and $v_{\mathcal{R}}$, to work with in trying to satisfy the incentive conditions (**??**) and (**??**) with $k_r^n = 1$ and $k_l^n = 0$ for majority voters. Our main result in this section relies on a construction that first ensures the dominant pivotal event to be $v = v_{\mathcal{R}}$ when the state is R and $v = v_{\mathcal{L}}$ when the state is L. This requires that

$$(1-\alpha)(1-q_l) + \alpha\overline{\theta} < (1-\alpha)q_r + \alpha\underline{\theta}. \tag{12}$$

The above condition requires that the maximum vote share for candidate $\mathcal{R}$ in state L be smaller than the minimum vote share for $\mathcal{R}$ in state R under sincere voting by all informed voters. It is satisfied when the size of aggregate uncertainty, $\alpha(\overline{\theta} - \underline{\theta})$, is sufficiently small, or when the signals precision, $q_r + q_l$, is sufficiently high. The rest of the construction uses appropriate choice of the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ to make the probabilities of the two dominant pivotal events, $g^n(v_{\mathcal{R}}|R)$ and $g^n(v_{\mathcal{L}}|L)$, go to 0 at the same rate, so that the ratio $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ goes to a finite and positive limit. This limiting value of $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ can satisfy both inequalities (**??**) and (**??**) with slack, so there is an interval of the common prior belief μ for which sincere voting by all informed majority voters is incentive compatible. Since this construction depends only on the incentive to save recount costs, it is preference-independent and will work for informed minority votes as well. Thus sincere voting by all informed voters can be supported as an equilibrium.

Proposition 1. *Suppose that condition (**??**) holds. There exist $\underline{\mu}$ and $\overline{\mu}$ such that for any prior belief $\mu \in (\underline{\mu}, \overline{\mu})$, there are election rules $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ under which sincere voting by all informed voters is an equilibrium for sufficiently large n.*

Proof. Suppose that all informed voters vote sincerely so that the strategy profile is given by κ_T. Note that $z(L, \overline{\theta}; \kappa_T)$ is equal to the left-hand-side of (**??**) while $z(R, \underline{\theta}; \kappa_T)$ is equal to the right-hand-side. For any thresholds $v_{\mathcal{L}}$, $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$ that satisfy

$$z(L, \overline{\theta}; \kappa_T) < v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}} < z(R, \underline{\theta}; \kappa_T),$$

Lemma **??** and the ranking of convergence rates imply that

$$\lim_{n\to\infty} \frac{g^n(v_{\mathcal{C}}|R)}{g^n(v_{\mathcal{R}}|R)} = \lim_{n\to\infty} \frac{g^n(v_{\mathcal{L}}|R)}{g^n(v_{\mathcal{R}}|R)} = \lim_{n\to\infty} \frac{g^n(v_{\mathcal{C}}|L)}{g^n(v_{\mathcal{L}}|L)} = \lim_{n\to\infty} \frac{g^n(v_{\mathcal{R}}|L)}{g^n(v_{\mathcal{L}}|L)} = 0.$$

Furthermore,

$$\lim_{n\to\infty} \frac{g^n(v_{\mathcal{R}}|R)}{g^n(v_{\mathcal{L}}|L)} = \frac{f(\underline{\theta})}{f(\overline{\theta})} \sqrt{\frac{v_{\mathcal{L}}(1-v_{\mathcal{L}})}{v_{\mathcal{R}}(1-v_{\mathcal{R}})}} \lim_{n\to\infty} \left(\frac{I(v_{\mathcal{R}}; z(R,\underline{\theta};\kappa_T))}{I(v_{\mathcal{L}}; z(L,\overline{\theta};\kappa_T))} \right)^n.$$

When $v_{\mathcal{R}} = z(R,\underline{\theta};\kappa_T)$ and $v_{\mathcal{L}} = z(L,\overline{\theta};\kappa_T)$, the rates of convergence $I(v_{\mathcal{R}}; z(R,\underline{\theta};\kappa_T))$ and $I(v_{\mathcal{L}}; z(L,\overline{\theta};\kappa))$ are the same. Further, since $I(v_{\mathcal{R}}; z(R,\underline{\theta};\kappa_T))$ decreases continuously when $v_{\mathcal{R}}$ decreases from $z(R,\underline{\theta};\kappa_T)$ and since $I(v_{\mathcal{L}}; z(L,\overline{\theta};\kappa_T))$ decreases continuously when $v_{\mathcal{L}}$ increases from $z(L,\overline{\theta};\kappa_T)$, there exists a unique value v^* such that, for every $v^*_{\mathcal{L}} \in [z(L,\overline{\theta};\kappa_T), v^*)$, equation (**??**) implicitly defines $v^*_{\mathcal{R}} \in (v^*, z(R,\underline{\theta};\kappa_T)]$ such that the rates of convergence are the same. Pick any such pair of $v^*_{\mathcal{L}}$ and $v^*_{\mathcal{R}}$ with the restriction that $v^*_{\mathcal{L}} < v_{\mathcal{C}} < v^*_{\mathcal{R}}$, and denote for such a pair

$$\gamma^* \equiv \frac{f(\underline{\theta})}{f(\overline{\theta})} \sqrt{\frac{v^*_{\mathcal{L}}(1-v^*_{\mathcal{L}})}{v^*_{\mathcal{R}}(1-v^*_{\mathcal{R}})}}.$$

Then $g^n(v^*_{\mathcal{R}}|R)/g^n(v^*_{\mathcal{L}}|L)$ converges to the positive and finite value γ^* as n grows.

For such a pair $v^*_{\mathcal{L}}$ and $v^*_{\mathcal{R}}$ and for large n, the incentive conditions (**??**) and (**??**) for sincere voting by the majority voters become:

$$\frac{1-P_r}{P_r} < \gamma^* < \frac{1-P_l}{P_l}. \tag{13}$$

If the prior belief μ is equal to $1/(1+\gamma^*)$, then the above condition can be written as

$$\gamma^* \frac{1-q_l}{q_r} < \gamma^* < \gamma^* \frac{q_l}{1-q_r},$$

which is satisfied with slack because $q_r > 1 - q_l$. Since these inequalities hold with slack, they continue to hold for some interval of prior belief μ around $1/(1+\gamma^*)$.

Furthermore, given this choice of election rule, the incentive conditions (**??**) and (**??**) for sincere voting by the minority voters also reduce to the same condition (**??**) as n becomes large, because only the pivotal probabilities $g^n(v^*_{\mathcal{R}}|R)$ and $g^n(v^*_{\mathcal{L}}|L)$ matter in the limit and minority and majority voters have the same incentive to save recount cost. Thus, for n large, the election rule that induces sincerely voting by majority voters also induces sincerely voting by minority voters. ■

By inducing sincere voting for all informed voters, our election rule with recount has a better chance of achieving information efficiency than a standard election rule. Given the presence of the aggregate uncertainty, we adapt the concept of full information equivalence introduced in Feddersen and Pesendorfer (1997) to the present model.

Definition 1. *An election rule attains asymptotic information efficiency if for any $\epsilon > 0$, there is an N such that, for all $n \geq N$, there exists an equilibrium where, for any realization of the aggregate uncertainty state θ, candidate $\mathcal{L}$ is chosen in state L and candidate $\mathcal{R}$ is chosen in state R with probabilities greater than $1 - \epsilon$.*

It is easy to see that the sincere-voting equilibrium constructed in Proposition 1 achieves asymptotic efficiency. Under sincere voting by all informed voters, the law of large numbers implies that as n grows, in payoff state S and aggregate state θ, the distribution of the vote share for candidate $\mathcal{R}$ converges to $z(S, \theta; \kappa_T)$. Since $z(R, \theta; \kappa_T) > z(R, \underline{\theta}; \kappa_T) > v_{\mathcal{R}}$, candidate $\mathcal{R}$ will be chosen as the outright winner in state R with probability arbitrarily close to one in a large election for any realization of the aggregate uncertainty state. Similarly, since $z(L, \theta; \kappa_T) < z(L, \overline{\theta}; \kappa_T) < v_{\mathcal{L}}$, candidate $\mathcal{L}$ will be the outright winner in state L with probability close to one regardless of θ.

In contrast, even when condition (**??**) holds, there are election environments in which asymptotic information efficiency fails under a standard election rule without recounting. There are two separate aspects to this claim, one that has to do with the presence of aggregate uncertainty, and the other that has to do with conflicting preferences. The first is established in Feddersen and Pesendorfer (1997); here we adapt the claim to our specific model. The second is first pointed out in Bhattacharya (2008); we repeat it here to highlight our contribution of achieving information efficiency through recounting rules.

First, we consider an environment with aggregate uncertainty ($\alpha > 0$) but no conflicting preferences ($\beta = 0$). Define

$$\lambda \equiv \max_{\gamma \geq 0} \quad H\left(\frac{\mu q_r \gamma}{\mu q_r \gamma + (1-\mu)(1-q_l)}\right) - H\left(\frac{\mu(1-q_r)\gamma}{\mu(1-q_r)\gamma + (1-\mu) q_l}\right).$$

This turns out to be an upper-bound on the fraction of informed voters that cast their votes according to their private signal under any standard election rule. Since

H has a positive density over $[0,1]$, we have $\lambda < 1$.[8] A sufficient condition in our environment for the non-existence of a standard election rule that achieves asymptotic information efficiency is then

$$(1-\alpha)(1-q_l)\lambda + \alpha\overline{\theta} > (1-\alpha)q_r\lambda + \alpha\underline{\theta}, \tag{14}$$

which means that the largest vote share for candidate $\mathcal{R}$ in state L exceeds the smallest share for $\mathcal{R}$ in R. Note that (**??**) can hold without violating (**??**). When that is the case, asymptotic information efficiency is infeasible with a standard election rule but is attainable with a suitably chosen election rule with recount.

To establish the claim, fix any threshold $v_{\mathcal{C}}$ and consider any sequence of equilibria $\{k^n\} = \{(k_r^n, k_l^n)\}$. By the incentive conditions (**??**) and (**??**), with $\delta = 0$, voting sincerely is optimal for an informed voter with preference intensity t if and only if

$$\frac{1-P_r}{P_r} \leq \frac{1-t}{t}\frac{g^n(v_{\mathcal{C}}|R)}{g^n(v_{\mathcal{C}}|L)} \leq \frac{1-P_l}{P_l}.$$

Using the expressions for P_r and P_l we can further rewrite the above as

$$k_l^n = \frac{\mu(1-q_r)\gamma^n}{\mu(1-q_r)\gamma^n + (1-\mu)q_l} \leq t \leq \frac{\mu q_r \gamma^n}{\mu q_r \gamma^n + (1-\mu)(1-q_l)} = k_r^n,$$

where $\gamma^n = g^n(v_{\mathcal{C}}|R)/g^n(v_{\mathcal{C}}|L)$. Consider any positive and finite limit point γ of the sequence $\{\gamma^n\}$.[9] The law of large numbers implies that as n goes to infinity in the corresponding subsequence, in payoff state R and aggregate uncertainty state θ the vote share for candidate $\mathcal{R}$ converges to

$$z(R,\theta;k) = (1-\alpha)[q_r H(k_r) + (1-q_r)H(k_l)] + \alpha\theta,$$

where $k = (k_r, k_l)$ is the pair of preference thresholds corresponding to the limit γ. Asymptotic information efficiency requires the lowest value of the above expression, corresponding to $z(R,\underline{\theta};k)$, to be greater than or equal to $v_{\mathcal{C}}$. Symmetrically, in the same subsequence of equilibria, in state L the largest vote share for $\mathcal{R}$ needs to be smaller than or equal to $v_{\mathcal{C}}$:

$$z(L,\overline{\theta};k) = (1-\alpha)[(1-q_l)H(k_r) + q_l H(k_l)] + \alpha\overline{\theta} \leq v_{\mathcal{C}}.$$

[8]For example, if H is uniform on $[0,1]$ and $q_r = q_l = q$, then $\lambda = 2q - 1$.

[9]If the sequence $\{\gamma^n\}$ has a limit point that is either 0 or infinity, then the fraction of informed voters that vote sincerely vanishes in the limit, and the standard election rule given by $v_{\mathcal{C}}$ clearly does not attain asymptotic efficiency under aggregate uncertainty.

Since $q_r + q_l > 1$ and by definition $H(k_r) - H(k_l) \leq \lambda$, the above two requirements cannot be satisfied simultaneously under condition (**??**).

Second, we consider an environment with conflicting preferences ($\beta > 0$) but no aggregate uncertainty ($\alpha = 0$). Fix any threshold $v_{\mathcal{C}}$ and consider any sequence of equilibria. Again, it is convenient to work with the likelihood ratio $\gamma^n = g^n(v_{\mathcal{C}}|R)/g^n(v_{\mathcal{C}}|L)$. For a limit point of the sequence to achieve asymptotic efficiency, the corresponding limit point γ of γ^n must be finite and positive. Given this, we can use the incentive conditions (**??**) to (**??**), with $\delta = 0$, to derive the limit preference threshold types, $\kappa = ((k_r, k_l), (\tilde{k}_r, \tilde{k}_l))$, all as functions of γ. Then, we have a limit point γ of an equilibrium sequence γ^n if the counterpart to the equal-rate condition (**??**) holds:

$$I(v_{\mathcal{C}}; z(R;\kappa)) = I(v_{\mathcal{C}}; z(L;\kappa))$$

where the vote shares $z(R;\kappa)$ and $z(L;\kappa)$ are functions of γ through (**??**) and (**??**), because κ depends on γ. This limit point γ represents an asymptotically information efficient outcome if and only if $z(L;\kappa) < v_{\mathcal{C}} < z(R;\kappa)$. Now, by (**??**) and (**??**), the sign of $z(R;\kappa) - z(L;\kappa)$ is the same as

$$(1-\beta)[H(k_r) - H(k_l)] - \beta[\tilde{H}(\tilde{k}_r) - \tilde{H}(\tilde{k}_l)].$$

It is easy to see from the incentive conditions (**??**) to (**??**) that $k_r > k_l$ and $\tilde{k}_r > \tilde{k}_l$, and thus whether or not $z(L;\kappa) < z(R;\kappa)$, and hence whether or not asymptotic efficiency is achievable in a standard election, depends on the preference distributions H and $\tilde{H}$, as well as β. In contrast, condition (**??**) always holds when $\alpha = 0$, and thus asymptotic efficiency is achievable in elections with recount, regardless of H, $\tilde{H}$, and β.

4. Informative Voting

The sincere voting equilibrium result of Proposition 1 is in striking contrast with the well known result that under standard election rules voting according to one's own private information is generally inconsistent with the pivotal voting logic of strategic voting models (Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1996). Yet the scope of our result is limited as it requires that the prior belief μ to lie in some interval $(\underline{\mu}, \overline{\mu})$. Further, while there is a continuum of pairs $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ under which sincere voting obtains in equilibrium, for a given value of $v_{\mathcal{L}}$ there is at most one value of $v_{\mathcal{R}}$ satisfying the equal-rate condition to induce sincere voting. If either restriction is violated sincere voting is not an equilibrium for any finite n, and the question remains whether election rules with recounting improve on standard election rules with respect to information efficiency.

In this section we abandon the focus on sincere voting to study "informative voting" equilibria, in which only a subset of all informed voters vote according to the realization of their private signals. The main result of this section shows that whenever asymptotic information efficiency would obtain under sincere voting, an informative voting equilibrium exists which is asymptotically information efficient. Thus, even if the common prior belief μ is extreme and the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are not calibrated to satisfy the equal-rate condition (**??**), asymptotic information efficiency obtains though not all informed voters vote according to their signals.

For an election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ to achieve asymptotic information efficiency, a sufficient condition is that:

$$(1-\alpha)(1-q_l) + \alpha\overline{\theta} < v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}} < (1-\alpha)q_r + \alpha\underline{\theta}. \tag{15}$$

Unless otherwise specified we maintain this assumption throughout the remainder of this section. To ensure that the candidate matching the state is elected with probability arbitrarily close to one given such an election rule, the sequence of equilibrium strategy $\{\kappa^n\}$ must be such that

$$z(L, \overline{\theta}; \kappa^n) < v_{\mathcal{L}} < v_{\mathcal{R}} < z(R, \underline{\theta}; \kappa^n) \tag{16}$$

for all n sufficiently large. Thus, we proceed by showing that an equilibrium satisfying this property can be constructed.

While a strategy profile κ^n is in general a four dimensional object, we can show that for asymptotically information efficient sequences of equilibria at most two of the thresholds can be interior. This property substantially simplifies our equilibrium construction.

Lemma 2. *Let $\{\kappa^n\}$ be a sequence of equilibria such that (**??**) holds. For all n sufficiently large: (i) if either $k_r^n \in (0,1)$ or $\tilde{k}_r^n \in (0,1)$, then $k_l^n = 0$ and $\tilde{k}_l^n = 1$; (ii) if either $k_l^n \in (0,1)$ or $\tilde{k}_l^n \in (0,1)$, then $k_r^n = 1$ and $\tilde{k}_r^n = 0$.*

Proof. By Lemma **??**, condition (**??**) implies that the likelihood ratios $g^n(v|L)/g^n(v_{\mathcal{L}}|L)$ for $v = v_{\mathcal{C}}, v_{\mathcal{R}}$ and $g^n(v|L)/g^n(v_{\mathcal{R}}|R)$ for $v = v_{\mathcal{L}}, v_{\mathcal{C}}$ become arbitrarily close to zero as n grows arbitrarily large. Suppose $k_r^n \in (0,1)$. The incentive condition (**??**) for majority voters with preference intensity k_r^n and signal r holds as equality, thus $(1-P_r)/P_r - g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ is arbitrarily close to zero. This in turn implies that $(1-P_l)/P_l - g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ is bounded above zero because P_r is bounded above P_l. For majority holders who observe signal l, since all other pivotal probabilities in the incentive condition (**??**) are dominated by either $g^n(v_{\mathcal{R}}|R)$ or $g^n(v_{\mathcal{L}}|L)$, condition (**??**) must hold as a strict inequality for any $t \in [0,1]$. Thus, all of them vote for candidate $\mathcal{L}$, meaning that $k_l^n = 0$. Likewise, the incentive condition (**??**) for minority voters who observe signal l to vote for candidate $\mathcal{L}$ is also satisfied as a strictly inequality, meaning that $\tilde{k}_l^n = 1$. This establishes the lemma for the case when $k_r^n \in (0,1)$. A similar argument applies to the case when $\tilde{k}_r^n$, k_l^n, or $\tilde{k}_l^n$ is strictly between 0 and 1. ∎

The logic of Lemma **??** can be intuitively explained. In an asymptotically efficient equilibrium the event $v = v_{\mathcal{L}}$ in state L and the event $v = v_{\mathcal{R}}$ in state R dominate all other pivotal events. Since at these pivotal events informed voters strictly prefer voting for one candidate or the other independently of their preference type, a voter can be indifferent between voting for either candidate only if these two events have probabilities arbitrarily close to each other. Since the private belief about the payoff state changes discretely with a voter's private signal, if a voter is indifferent between the two candidates after observing one signal, then regardless of his preference type the voter will strictly prefer one candidate over the other when observing the opposite signal.

We have seen that in a standard election with a single pivotal event of $v = v_{\mathcal{C}}$, the ratio $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ must go to a positive a finite limit in equilibrium for large elections for asymptotic information efficiency to obtain. This implies that the equilibrium threshold types are strictly between zero and one, which means that the fraction of informed voters who vote according to their signals is always less than one. By introducing the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$, Lemma **??** allows us to construct "semi-sincere" equilibria in which either (i) all informed voters vote for candidate $\mathcal{L}$ when they observe signal l; or (ii) all informed voters vote for candidate $\mathcal{R}$ when they observe signal r.

It is easy to use the vote share functions to show that it is impossible for any interior equilibrium κ' to aggregate more information about the states than a "semi-sincere" equilibrium κ does for any realization of aggregate uncertainty. Suppose by way of contradiction that this is not true, so that for some θ, both $z(R,\theta;\kappa') > z(R,\theta;\kappa)$ and $z(L,\theta;\kappa') < z(L,\theta;\kappa)$. Using the definition of the vote share equations (**??**) and (**??**) and the fact that $q_r > 1 - q_l$, these two inequalities imply

$$\begin{aligned}(1-\beta)[H(k'_r) - H(k_r)] - \beta[\tilde{H}(\tilde{k}'_r) - \tilde{H}(\tilde{k}_r)] \\ > (1-\beta)[H(k'_l) - H(k_l)] - \beta[\tilde{H}(\tilde{k}'_l) - \tilde{H}(\tilde{k}_l)]. \end{aligned} \tag{17}$$

If $k_l = 0$ and $\tilde{k}_l = 1$, the left-hand-side of (**??**) must be positive, which then contradicts $z(L,\theta;\kappa') < z(L,\theta;\kappa)$ because $k'_l > 0$ and $\tilde{k}'_l < 1$. If $k_r = 1$ and $\tilde{k}_r = 0$, then the right-hand-side of (**??**) must be negative, which contradicts $z(R,\theta;\kappa') > z(R,\theta;\kappa)$ because $k'_r < 1$ and $\tilde{k}'_r > 0$.[10]

Given Lemma **??** we proceed to construct an informationally efficient equilibrium which exhibits "semi-sincere" voting. For any given election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ that satisfies (**??**), there are three possibilities. First, if

$$I(v_{\mathcal{L}}; z(L,\overline{\theta};\kappa_T)) > I(v_{\mathcal{R}}; z(R,\underline{\theta};\kappa_T)), \tag{18}$$

then sincere voting cannot be supported as an equilibrium. But we can construct an informative voting equilibrium in which all informed voters vote $\mathcal{L}$ after observing an l private signal, but only some of the voters vote for $\mathcal{R}$ when observing an r signal. In the second case, if the opposite inequality holds, then an informative equilibrium can be constructed when a private signal r always induces a vote for $\mathcal{R}$, while an l signal leads to a vote for $\mathcal{L}$ only for a subset of all preference types. Of course, the final cases in which the two rates are equal corresponds to the case studied in Section 3, and a sincere voting equilibrium can be constructed.

In the remainder of this section we study the first case where (**??**) holds and construct an equilibrium with informed voters always voting for $\mathcal{L}$ upon observing signal l. The construction for the second case is symmetric. For any strategy κ^n with $k^n_l = 0$ and $\tilde{k}^n_l = 1$,

$$z(L,\overline{\theta};\kappa^n) = (1-\alpha)(1-q_l)\left[(1-\beta)H(k^n_r) + \beta(1-\tilde{H}(\tilde{k}^n_r))\right] + \alpha\overline{\theta}.$$

[10]Without restrictions on preference distributions H and $\tilde{H}$, it is not generally true that $z(R,\theta;\kappa) - z(L,\theta;\kappa)$ always exceeds $z(R,\theta;\kappa') - z(L,\theta;\kappa')$.

Since the right-hand-side of the above equation is no larger than $(1-\alpha)(1-q_l) + \alpha\overline{\theta}$, the asymptotic information efficiency condition in state L, $z(L, \overline{\theta}; \kappa^n) < v_{\mathcal{L}}$, is satisfied for any pair k_r^n and $\tilde{k}_r^n$. The asymptotic efficiency condition in state R is:

$$z(R, \underline{\theta}; \kappa^n) = (1-\alpha)q_r \left[(1-\beta)H(k_r^n) + \beta(1-\tilde{H}(\tilde{k}_r^n))\right] + \alpha\underline{\theta} > v_{\mathcal{R}}. \tag{19}$$

Our main result in this section establishes the existence of a sequence of informative voting equilibria that achieves asymptotic efficiency. The proof of the following proposition is constructive. We derive the best responses of both majority and minority voters assuming that the strategy profile satisfies the asymptotic efficiency condition (**??**), and then show that there is a unique fixed point of these best responses.

Proposition 2. *Suppose that the election rule satisfies (**??**). There exists a unique sequence of equilibria that achieves asymptotic information efficiency.*

Proof. We consider only the first case where (**??**) holds. The opposite case is symmetric.

For any strategy κ^n, fix $k_l^n = 0$ and $\tilde{k}_l^n = 1$. From condition (**??**) and the definition of $z(R, \underline{\theta}; \kappa^n)$,

$$z\left(R, \underline{\theta}; ((1, k_l^n), (0, \tilde{k}_l^n))\right) = (1-\alpha)q_r + \alpha\underline{\theta} > v_{\mathcal{R}} > \alpha\underline{\theta} = z\left(R, \underline{\theta}; ((0, k_l^n), (1, \tilde{k}_l^n))\right).$$

Since $z(R; \underline{\theta}; \cdot)$ is strictly increasing in k_r^n, strictly decreasing in $\tilde{k}_r^n$, and continuous in both argument, we can define a continuous and non-decreasing function $Y^n(\tilde{k}_r^n)$: $[0,1] \to [0,1]$ such that:

$$Y^n(\tilde{k}_r^n) = \begin{cases} 1 & \text{if } z(R, \underline{\theta}; ((1,0), (\tilde{k}_r^n, 1))) \geq v_{\mathcal{R}}, \\ 0 & \text{if } z(R, \underline{\theta}; ((0,0), (\tilde{k}_r^n, 1))) \leq v_{\mathcal{R}}, \\ \{k_r^n : z(R, \underline{\theta}; ((k_r^n, 0), (\tilde{k}_r^n, 1))) = v_{\mathcal{R}}\} & \text{otherwise.} \end{cases}$$

The set of strategy profiles k_r^n and $\tilde{k}_r^n$ such that $k_r^n > Y^n(\tilde{k}_r^n)$ satisfies the condition (**??**) for asymptotic efficiency.

Consider the difference in the expected payoff upon observing signal r between voting for $\mathcal{R}$ and voting for $\mathcal{L}$ for the majority threshold type k_r^n:

$$\begin{aligned} D^n(k_r^n, \tilde{k}_r^n) \equiv\ & P_r[g^n(v_{\mathcal{R}}|R)\delta + g^n(v_{\mathcal{C}}|R)(1-k_r^n) - g^n(v_{\mathcal{L}}|R)\delta] \\ & - (1-P_r)[g^n(v_{\mathcal{L}}|L)\delta + g^n(v_{\mathcal{C}}|L)k_r^n - g^n(v_{\mathcal{R}}|L)\delta], \end{aligned}$$

where $g^n(\cdot|\cdot)$ depends on the k_r^n and $\tilde{k}_r^n$. We claim that in the region where $z(R,\underline{\theta};\kappa^n) > v_{\mathcal{R}}$, or equivalently $k_r^n > Y^n(\tilde{k}_r^n)$, the utility difference D^n is strictly decreasing in k_r^n and strictly increasing in $\tilde{k}_r^n$ for n large enough.

To see this, note that k_r^n increases both $z(R,\theta;\kappa^n)$ and $z(L,\theta;\kappa^n)$ for all θ. Furthermore, from formula (**??**) for binomial probability, we obtain

$$\frac{\partial g^n(v|S,\theta)}{\partial k_r^n} = ng^n(v|S,\theta)\frac{v - z(S,\theta;\kappa^n)}{z(S,\theta;\kappa^n)(1-z(S,\theta;\kappa^n))}\frac{\partial z(S,\theta;\kappa^n)}{\partial k_r^n}.$$

Thus, by the same argument as in Lemma **??**,

$$\lim_{n\to\infty}\frac{\partial g^n(v|S)/\partial k_r^n}{\partial g^n(v'|S')/\partial k_r^n} = 0 \quad \text{if} \quad \lim_{n\to\infty}\frac{g^n(v|S,\theta(v,S))}{g^n(v'|S',\theta(v',S'))} = 0.$$

Because $z(R,\underline{\theta};\kappa^n) > v_{\mathcal{R}}$, $\partial g^n(v|R)/\partial k_r^n$ is dominated by $\partial g^n(v_{\mathcal{R}}|R)/\partial k_r^n$ for $v = v_{\mathcal{L}}, v_{\mathcal{C}}$ as n grows large. Similarly, because $z(L,\overline{\theta};\kappa_n) \leq z(L,\overline{\theta};\kappa_T) < v_{\mathcal{L}}$, $\partial g^n(v|L)/\partial k_r^n$ is dominated by $\partial g^n(v_{\mathcal{L}}|L)/\partial k_r^n$ for $v = v_{\mathcal{C}}, v_{\mathcal{L}}$. For n sufficiently large, therefore, the derivative of D^n with respect to k_r^n is determined by

$$P_r\delta\frac{\partial g^n(v_{\mathcal{R}}|R)}{\partial k_r^n} - (1-P_r)\delta\frac{\partial g^n(v_{\mathcal{L}}|L)}{\partial k_r^n} - P_r g^n(v_{\mathcal{C}}|R) - (1-P_r)g^n(v_{\mathcal{L}}|L).$$

The first term is negative because $g^n(v_{\mathcal{R}}|R,\theta)$ is decreasing in $z(R,\theta;\kappa^n)$ for all θ, while the second term is positive because $g^n(v_{\mathcal{L}}|L,\theta)$ is increasing in $z(L,\theta;\kappa^n)$ for all θ. It follows that for large n, D^n is strictly decreasing in k_r^n for all $k_r^n > Y^n(\tilde{k}_r^n)$.

A similar argument establishes that for large n, the utility difference D^n is strictly increasing in $\tilde{k}_r^n$ for all $\tilde{k}_r^n$ such that $Y^n(\tilde{k}_r^n) < k_r^n$. Briefly, $\tilde{k}_r^n$ decreases both $z(R,\theta;\kappa^n)$ and $z(L,\theta;\kappa^n)$ for all θ. For n sufficiently large, the derivative of D^n with respect to $\tilde{k}_r^n$ is determined by

$$P_r\delta\frac{\partial g^n(v_{\mathcal{R}}|R)}{\partial \tilde{k}_r^n} - (1-P_r)\delta\frac{\partial g^n(v_{\mathcal{L}}|L)}{\partial \tilde{k}_r^n}.$$

The first term is positive because $g^n(v_{\mathcal{R}}|R,\theta)$ is decreasing in $z(R,\theta;\kappa^n)$ for all θ, while the second term is negative because $g^n(v_{\mathcal{L}}|L,\theta)$ is increasing in $z(L,\theta;\kappa^n)$ for all θ.

Given the above results, we define a best response $B^n(\tilde{k}_r^n)$ by majority voters to the strategy $\tilde{k}_r^n$ of minority voters for sufficiently large n:

$$B^n(\tilde{k}_r^n) = \begin{cases} 1 & \text{if } D^n(1, \tilde{k}_r^n) \geq 0, \\ 0 & \text{if } D^n(Y^n(\tilde{k}_r^n), \tilde{k}_r^n) \leq 0, \\ \{k_r^n \in (Y^n(\tilde{k}_r^n), 1) : D^n(k_r^n, \tilde{k}_r^n) = 0\} & \text{otherwise.} \end{cases}$$

In the first case, because $D^n(\cdot, \tilde{k}_r^n)$ is decreasing for $k_r^n \geq Y^n(\tilde{k}_r^n)$, if $D^n(1, \tilde{k}_r^n) \geq 0$ then $D^n(k_r^n, \tilde{k}_r^n) > 0$ for all $k_r^n \in (Y^n(\tilde{k}_r^n), 1)$, in which case it is a best response for all majority voters to vote for $\mathcal{R}$ upon observing signal r. In the second case, if $Y^n(\tilde{k}_r^n) > 0$, then by definition we have $z(R, \underline{\theta}; \kappa^n) \geq v_{\mathcal{R}}$ when the strategy κ^n is such that $k_r^n = Y^n(\tilde{k}_r^n)$. For such a strategy κ^n, we have

$$I\left(v_{\mathcal{R}}; z(R, \underline{\theta}; \kappa^n)\right) = 1 > I\left(v_{\mathcal{L}}; z(L, \overline{\theta}; \kappa^n)\right),$$

and thus $D^n(Y^n(\tilde{k}_r^n), \tilde{k}_r^n)$ must be strictly positive by Lemma **??**. Therefore the utility difference $D^n(Y^n(\tilde{k}_r^n), \tilde{k}_r^n)$ can be non-positive only when $Y^n(\tilde{k}_r^n) = 0$, in which case $D^n(Y^n(\tilde{k}_r^n), \tilde{k}_r^n) \leq 0$ implies $D^n(k_r^n, \tilde{k}_r^n) < 0$ for all $k_r^n > 0$, which means that no majority voter would vote for $\mathcal{R}$ upon observing signal r. In the third case, neither $D^n(1, \tilde{k}_r^n) \geq 0$ nor $D^n(Y^n(\tilde{k}_r^n), \tilde{k}_r^n) \leq 0$ is true. There must be a unique $k_r^n \in (Y^n(\tilde{k}_r^n), 1))$ that solves the indifference condition $D^n(k_r^n, \tilde{k}_r^n) = 0$, and such k_r^n defines the threshold type of majority voters who vote for $\mathcal{R}$ upon observing signal r. Since $D^n(k_r^n, \tilde{k}_r^n)$ is strictly increasing and continuous in $\tilde{k}_r^n$, the best-response function $B^n(\tilde{k}_r^n) : [0, 1] \to [0, 1]$ is non-decreasing and continuous. Furthermore, since by assumption the inequality (**??**) holds, $D^n(1, 0) < 0$. Thus we must have $B^n(0) < 1$.

Similarly, consider the difference in the expected payoff upon observing r between voting for $\mathcal{R}$ and voting for $\mathcal{L}$ for the minority threshold type $\tilde{k}_r^n$:

$$\begin{aligned} \tilde{D}^n(k_r^n, \tilde{k}_r^n) \equiv\ & P_r[g^n(v_{\mathcal{R}}|R)\delta - g^n(v_{\mathcal{C}}|R)\tilde{k}_r^n - g^n(v_{\mathcal{L}}|R)\delta] \\ & - (1 - P_r)[g^n(v_{\mathcal{L}}|L)\delta - g^n(v_{\mathcal{C}}|L)(1 - \tilde{k}_r^n) - g^n(v_{\mathcal{R}}|L)\delta]. \end{aligned}$$

We can obtain a non-decreasing, continuous best-response function for the minority threshold type $\tilde{B}^n(k_r^n) : [0, 1] \to [0, 1]$, such that for sufficiently large n: (i) $Y(\tilde{B}^n(k_r^n)) \leq k_r^n$, with strict inequality if $Y(\tilde{B}^n(k_r^n)) \in (0, 1)$; (ii) $\tilde{D}^n(k_r^n, \tilde{B}^n(k_r^n)) = 0$ if $\tilde{B}^n(k_r^n) \in (0, 1)$; and (iii) $\tilde{B}^n(1) > 0$.

A standard fixed point argument then implies that, for all n sufficiently large, there exists a pair of thresholds $(k_r^n, \tilde{k}_r^n)$ that is an intersection of B^n and $\tilde{B}^n$. At any intersection, we have $k_r^n \geq Y(\tilde{k}_r^n)$ by the construction of B^n and $\tilde{B}^n$, with strict inequality unless $k_r^n = \tilde{k}_r^n = 1$ or $k_r^n = \tilde{k}_r^n = 0$, so that the asymptotic efficiency condition (**??**) is satisfied. It remains to argue that any intersection $(k_r^n, \tilde{k}_r^n)$ of B^n and $\tilde{B}^n$, together with $k_l^n = 0$ and $\tilde{k}_l^n = 1$, constitutes an equilibrium. Note that by construction, majority and minority voters are playing best responses after receiving their private signal r. If either k_r^n or $\tilde{k}_r^n$ is interior, then by Lemma **??**, $k_l^n = 0$ and $\tilde{k}_l^n = 1$ are also best responses for majority and minority voters after receiving their private signal l for n sufficiently large. If $k_r^n = \tilde{k}_r^n = 1$, then the fact that this pair is a fixed point of the best-response mapping implies that $D^n(1,1) \geq 0$ and $\tilde{D}^n(1,1) \leq 0$. The condition $D^n(1,1) \geq 0$ can be written as:

$$\frac{g^n(v_{\mathcal{R}}|R)\delta - g^n(v_{\mathcal{L}}|R)\delta}{g^n(v_{\mathcal{L}}|L)\delta + g^n(v_{\mathcal{C}}|L) - g^n(v_{\mathcal{R}}|L)\delta} \geq \frac{1-P_r}{P_r}.$$

Likewise, the condition $\tilde{D}^n(1,1) \leq 0$ can be written as:

$$\frac{g^n(v_{\mathcal{R}}|R)\delta - g^n(v_{\mathcal{C}}|R) - g^n(v_{\mathcal{L}}|R)\delta}{g^n(v_{\mathcal{L}}|L)\delta - g^n(v_{\mathcal{R}}|L)\delta} \leq \frac{1-P_r}{P_r}.$$

By Lemma **??**, the left-hand-side expressions of the two inequalities above are both arbitrarily close to the ratio $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$, which implies that the same ratio is arbitrarily close to $(1-P_r)/P_r$, and is therefore strictly less than $(1-P_l)/P_l$. Thus, $k_l^n = 0$ and $\tilde{k}_l^n = 1$ are best responses for majority and minority voters after receiving their private signal l. Finally, if $k_r^n = \tilde{k}_r^n = 0$, then it must be the case that $D^n(0,0) \leq 0$ and $\tilde{D}^n(0,0) \geq 0$. These two inequalities again imply that $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ is arbitrarily close to $(1-P_r)/P_r$, yielding the same conclusion that $k_l^n = 0$ and $\tilde{k}_l^n = 1$ are best responses for majority and minority voters after receiving their private signal l.

To prove the uniqueness claim we establish that for any n the functions B^n and $\tilde{B}^n$ have a unique intersection. A sufficient condition for this result is that at any interior intersection point the slope of B^n is strictly smaller than the inverse of the slope of $\tilde{B}^n$ so that in the $(k_r^n, \tilde{k}_r^n)$ space, $\tilde{B}^n$ crosses B^n at most once and from above. To see this, first note that for any v and S,

$$\frac{\partial g^n(v|S)}{\partial k_r^n} = -\frac{1-\beta}{\beta}\frac{h(k_r^n)}{\tilde{h}(\tilde{k}_r^n)}\frac{\partial g^n(v|S)}{\partial \tilde{k}_r^n}.$$

Define

$$
\begin{aligned}
M \equiv\ & P_r\delta\left(\frac{\partial g^n(v_{\mathcal{R}}|R)}{\partial \tilde{k}_r^n} - \frac{\partial g^n(v_{\mathcal{L}}|R)}{\partial \tilde{k}_r^n}\right) - (1-P_r)\delta\left(\frac{\partial g^n(v_{\mathcal{L}}|L)}{\partial \tilde{k}_r^n} - \frac{\partial g^n(v_{\mathcal{R}}|L)}{\partial \tilde{k}_r^n}\right)\\
&+ P_r\frac{\partial g^n(v_{\mathcal{C}}|R)}{\partial \tilde{k}_r^n}(1-k_r^n) - (1-P_r)\frac{\partial g^n(v_{\mathcal{C}}|L)}{\partial \tilde{k}_r^n}k_r^n;\\
\tilde{M} \equiv\ & P_r\delta\left(\frac{\partial g^n(v_{\mathcal{R}}|R)}{\partial \tilde{k}_r^n} - \frac{\partial g^n(v_{\mathcal{L}}|R)}{\partial \tilde{k}_r^n}\right) - (1-P_r)\delta\left(\frac{\partial g^n(v_{\mathcal{L}}|L)}{\partial \tilde{k}_r^n} - \frac{\partial g^n(v_{\mathcal{R}}|L)}{\partial \tilde{k}_r^n}\right)\\
&- P_r\frac{\partial g^n(v_{\mathcal{C}}|R)}{\partial \tilde{k}_r^n}(1-\tilde{k}_r^n) + (1-P_r)\frac{\partial g^n(v_{\mathcal{C}}|L)}{\partial \tilde{k}_r^n}\tilde{k}_r^n.
\end{aligned}
$$

Note that M and $\tilde{M}$ are strictly positive positive for n large enough. By differentiating the indifference conditions of the majority type k_r^n and the minority type $\tilde{k}_r^n$ we obtain

$$
\begin{aligned}
\frac{\partial B^n(\tilde{k}_r^n)}{\partial \tilde{k}_r^n} &= \frac{M}{M((1-\beta)/\beta)(h(k_r^n)/\tilde{h}(\tilde{k}_r^n)) + P_r g^n(v_{\mathcal{C}}|R) + (1-P_r)g^n(v_{\mathcal{C}}|L)}\\
&< \frac{\beta}{1-\beta}\frac{\tilde{h}(\tilde{k}_r^n)}{h(k_r^n)}\\
&< \frac{\tilde{M} + P_r g^n(v_{\mathcal{C}}|R) + (1-P_r)g^n(v_{\mathcal{C}}|L)}{\tilde{M}((1-\beta)/\beta)(h(k_r^n)/\tilde{h}(\tilde{k}_r^n))} = \left(\frac{\partial \tilde{B}^n(k_r^n)}{\partial k_r^n}\right)^{-1}. \qquad \blacksquare
\end{aligned}
$$

Proposition **??** establishes that, for large n, an informative voting equilibrium exists in which the pivotal event $v_{\mathcal{R}}$ dominates the others when the state is R and the pivotal event $v_{\mathcal{L}}$ dominates the others in state L. The equilibrium constructed in the proof has the property that voting is semi-sincere: one of the signal realizations is always reported sincerely, while the other signal is misreported by some types. Furthermore, as in the sincere voting equilibrium of Proposition **??**, any limit point κ^* of an asymptotic efficient equilibrium sequence $\{\kappa^n\}$ constructed in Proposition **??** satisfies a version of the equal-rate condition:

$$
I(v_{\mathcal{L}}; z(L, \overline{\theta}; \kappa^*)) = I(v_{\mathcal{R}}; z(R, \underline{\theta}; \kappa^*)). \tag{20}
$$

Otherwise, the ratio $g^n(v_{\mathcal{R}}|R)/g^n(v_{\mathcal{L}}|L)$ would be either arbitrarily small or arbitrarily large for all n sufficiently large. Then, regardless of their private signals, all informed voters strictly prefer voting for $\mathcal{L}$ or voting for $\mathcal{R}$, contradicting the assumed asymptotic efficiency of $\{\kappa^n\}$.

An important consequence of (**??**) is that sincere voting is a robust feature under our election rules with recounting, in the sense that if the recounting thresholds $v_{\mathcal{L}}$

and $v_{\mathcal{R}}$ are close to what induce sincere voting in the construction of Proposition **??**, then any limit point κ^* of an asymptotic efficient equilibrium sequence $\{\kappa^n\}$ in Proposition **??** is close to κ_T. More precisely, assume that $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are such that (**??**) holds, so that any limit point κ^* of an asymptotic efficient equilibrium sequence $\{\kappa^n\}$ in Proposition **??** has $k_l^* = 0$ and $\tilde{k}_l^* = 1$. We claim that for any $\epsilon > 0$ there exists an $\eta > 0$ such that if

$$\left| I(v_{\mathcal{L}}; z(L, \overline{\theta}; \kappa_T)) - I(v_{\mathcal{R}}; z(R, \underline{\theta}; \kappa_T)) \right| < \eta,$$

then $k_r^* > 1 - \epsilon$ and $\tilde{k}_r^* < \epsilon$. To see this, note that if two limit points of $(k_r^*, \tilde{k}_r^*)$ and $(k_r^{*\prime}, \tilde{k}_r^{*\prime})$ both satisfy the equal-rate condition (**??**) for the same recounting thresholds $v_{\mathcal{L}}, v_{\mathcal{R}}$, then $k_r^* > k_r^{*\prime}$ if and only if $\tilde{k}_r^* > \tilde{k}_r^{*\prime}$. This follows because $z(R, \underline{\theta}; \cdot)$ and $z(L, \overline{\theta}; \cdot)$ are strictly increasing in k_r and strictly decreasing in $\tilde{k}_r$, and because both limit points are asymptotic efficient, implying that $I(v_{\mathcal{L}}; \cdot)$ is strictly increasing in $z(L, \overline{\theta}; \cdot)$ and $I(v_{\mathcal{R}}; \cdot)$ strictly decreasing in $z(R, \underline{\theta}; \cdot)$. Since $I(v_{\mathcal{L}}; z(L, \overline{\theta}; ((\cdot, 0), (\cdot, 1))))$ and $I(v_{\mathcal{R}}; z(R, \underline{\theta}; ((\cdot, 0), (\cdot, 1))))$ are continuous in k_r and $\tilde{k}_r$, if $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are such that $I(v_{\mathcal{L}}; z(L, \overline{\theta}; ((1, 0), (0, 1))))$ and $I(v_{\mathcal{R}}; z(R, \underline{\theta}; ((1, 0), (0, 1))))$ are arbitrarily close to each other, then for any limit point κ^* of an asymptotic efficient equilibrium sequence $\{\kappa^n\}$, condition (**??**) implies that k_r^* and $\tilde{k}_r^*$ are arbitrarily close to 1 and 0 respectively.

The result that there is a sequence of equilibria that achieves asymptotic efficiency can be further strengthened. The sufficient condition (**??**) stated in Proposition **??** for this result requires that for the given election rule sincere voting would achieve asymptotic information efficiency. When the election rule is skewed toward one candidate, for example by requiring that his opponent is elected only if receiving a very large fraction of votes, (i.e., $v_{\mathcal{C}}$ very small or very large), sincere voting might not achieve asymptotic efficiency regardless of the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ and the amount of aggregate uncertainty. This happens if $v_{\mathcal{C}}$ is strictly smaller than $1 - q_l$ or strictly larger than q_r. In the first case, even without aggregate uncertainty, the expected vote share of candidate $\mathcal{R}$ under sincere voting is too large to lose the election in either state, the opposite is true in the second case. When sincere voting fails there could be other strategy profiles that achieve asymptotic efficiency. Given an election rule with recounting, a necessary and sufficient condition for a strategy profile κ to achieve asymptotic efficiency is that

$$z(L, \overline{\theta}; \kappa) < v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}} < z(R, \underline{\theta}; \kappa). \tag{21}$$

Proposition **??** can be strengthen to show that, given an arbitrary election rule with recounting, if for some strategy profile κ condition (**??**) is satisfied, then there exists a sequence of equilibria that achieves asymptotic information efficiency.[11] This result immediately implies that for any standard electoral rule $v_{\mathcal{C}}$ without recounting, if it is at all possible to achieve asymptotic information efficiency through any voting strategy (which need not be an equilibrium), then there are recounting thresholds such that information efficiency is achieved asymptotically along a sequence of equilibria. For sufficiently small aggregate uncertainty, asymptotic information efficiency can always be obtained with recounting for any value of $v_{\mathcal{C}}$ like in Feddersen and Pesendorfer (1997). This implies that asymptotic information efficiency can also be achieved under our election rule with the same $v_{\mathcal{C}}$ and recounting thresholds close to $v_{\mathcal{C}}$, even though the sufficient condition (**??**) does not hold.

5. Discussion

5.1. Two rounds of voting

Suppose the election rule is that candidate $\mathcal{R}$ is the outright winner if his vote share in the first round of voting is greater than $v_{\mathcal{R}}$, and candidate $\mathcal{L}$ is the outright winner if his vote share is greater than $1 - v_{\mathcal{L}}$. When neither candidate is an outright winner, there will be a second round of voting with a standard election rule $v_{\mathcal{C}}$ after imposing a second-round voting cost δ to each voter. We claim that under this alternative specification of the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$, the equilibrium construction for our model of election with recounting can be replicated as an equilibrium in a model with two rounds of voting.

The equilibrium construction in such a model with two voting rounds poses some additional complications. First, there is a continuum of pivotal events because any realized first-round vote share for $\mathcal{R}$ between $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ might in principle lead to a different continuation equilibrium. However, if the first round strategy profile

[11] The logic of this result is similar to that of Proposition **??**. After appropriately constraining the strategy profile so that one of the two signals is always voted sincerely, for all n large enough a non-empty region in the space of the remaining two thresholds can be identified such that (**??**) holds. Both the lower bound on $z(R, \underline{\theta}; \cdot)$ and the upper bound on $z(L, \overline{\theta}; \cdot)$ are will be binding somewhere along the boundary of this region, unlike the case in Proposition **??** where one of the two constraints is always slack. A construction similar to that in the proof of Proposition **??** can then be used to show that an equilibrium within this region exists for all n sufficiently large.

satisfies the information efficiency condition (**??**), then it follows from Lemma **??** that for n sufficiently large the only probabilistically relevant pivotal events in the first round are that $v = v_{\mathcal{L}}$ in state L and that $v = v_{\mathcal{R}}$ in state R. All other pivotal events are dominated by one of these two events. A second complication in replicating our equilibrium construction arises because, at the two relevant pivotal events, the vote of an informed voter will change the timing of the election resolution—as in the recounting model—but might also change the election outcome. However, if the first round strategy profile satisfies the information efficiency constraint (**??**), for n large the belief that the state is R is arbitrarily close to 1 at the pivotal event $v_{\mathcal{R}}$ and to 0 at the pivotal event $v_{\mathcal{L}}$. As long as in the continuation equilibrium the probability that $\mathcal{R}$ is elected approaches 1 (respectively, 0) when every informed voter's belief that the state is R is close to 1 (respectively, 0) then at the pivotal events $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ the vote affects the election outcome (i.e., which candidate wins) with a vanishing probability. In other words, the dominant consideration in the first round of voting is to avoid the cost δ incurred in a second round of voting, and our equilibrium construction for the recount model is replicated in a model with two rounds of voting.

5.2. Recount cost

Our model of election with recount does not depend on the magnitude of the recount cost δ. We only assume that δ is positive and fixed as n goes to infinity. This restriction can be further relaxed by assuming that a recount costs a fixed amount of $\Delta > 0$ and that in an election with $n+1$ voters, each voter bears a cost of $\delta^n = \Delta/(n+1)$.

When $z(R, \underline{\theta}; \kappa^n) > v_{\mathcal{R}}$, Lemma **??** implies that $g^n(v|R)/g^n(v_{\mathcal{R}}|R)$ goes to 0 as n goes to infinity for $v = v_{\mathcal{L}}, v_{\mathcal{C}}$. Moreover, the ratio goes to 0 at an exponential rate because the rate functions of the different pivotal events are ranked. On the other hand, even though the recount cost δ^n also goes to 0 as n goes to infinity, it goes to 0 only at the rate $1/n$. Consequently, we have

$$\lim_{n\to\infty} \frac{g^n(v_{\mathcal{R}}|R)\delta^n + g^n(v_{\mathcal{C}}|R)(1-t) - g^n(v_{\mathcal{L}}|R)\delta^n}{g^n(v_{\mathcal{L}}|L)\delta^n + g^n(v_{\mathcal{C}}|L)t - g^n(v_{\mathcal{R}}|L)\delta^n} = \lim_{n\to\infty} \frac{g^n(v_{\mathcal{R}}|R)}{g^n(v_{\mathcal{L}}|L)}$$

for majority voters, and the same is true for minority voters after substituting $-(1-\tilde{t})$ for $1-t$ and $-\tilde{t}$ for t. The remainder of the proofs of Propositions **??** and **??** goes through with no change.

5.3. Counting errors

Our model does not allow for counting errors, so that the vote count in the initial stage is identical to the vote count in the recount stage. There are different ways to introduce counting errors. We consider two alternatives.

In the first version of a model with counting error, we assume that each vote for candidate $\mathcal{R}$ has an independent probability $\zeta < 1/2$ of being miscounted as a vote for candidate $\mathcal{L}$, and likewise each vote for $\mathcal{L}$ has an independent probability ζ of being miscounted as a vote for $\mathcal{R}$. Further assume that if there is a recount, all the counting errors are corrected. Under these assumptions, if the true vote share for candidate $\mathcal{R}$ is v, the initial vote count for $\mathcal{R}$ will be

$$v_e = (1-\zeta)v + \zeta(1-v).$$

Note that $v_e > v$ if and only if $v < 1/2$, which is due to regression to the mean. Define

$$v'_{\mathcal{L}} \equiv \frac{v_{\mathcal{L}} - \zeta}{1-2\zeta}, \qquad v'_{\mathcal{R}} \equiv \frac{v_{\mathcal{R}} - \zeta}{1-2\zeta}.$$

Then, under the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$, the election would go into the recount stage if the true vote share v for $\mathcal{R}$ is between $v'_{\mathcal{L}}$ and $v'_{\mathcal{R}}$. In general, $v'_{\mathcal{L}}$ can be greater than or less than $v_{\mathcal{C}}$. But if $v'_{\mathcal{L}} > v_{\mathcal{C}}$, candidate $\mathcal{R}$ would always win whenever there is a recount. To avoid this kind of situation, we maintain the assumption that the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ is such that

$$v'_{\mathcal{L}} < v_{\mathcal{C}} < v'_{\mathcal{R}}. \tag{22}$$

For example, if $v_{\mathcal{C}} = 1/2$, then (**??**) is always satisfied.

Given assumption (**??**), there are again three pivotal events: (i) $v_e = v'_{\mathcal{R}}$; (ii) $v_e = v'_{\mathcal{L}}$; and (iii) $v = v_{\mathcal{C}}$. For a majority voter who observes signal r, given pivotal event (i), casting a vote for candidate $\mathcal{R}$ has a probability $1-\zeta$ of saving the recount cost δ (when his vote is correctly counted), but has a probability ζ of paying the recount cost δ (when his vote is miscounted). Thus, voting for $\mathcal{R}$ is preferred to voting for $\mathcal{L}$ if

$$\begin{aligned} &P_r[g^n(v_{\mathcal{R}}|R)(1-2\zeta)\delta + g^n(v_{\mathcal{C}}|R)(1-t) - g^n(v_{\mathcal{L}}|R)(1-2\zeta)\delta] \\ &\quad \geq (1-P_r)[g^n(v_{\mathcal{L}}|L)(1-2\zeta)\delta + g^n(v_{\mathcal{C}}|L)t - g^n(v_{\mathcal{R}}|L)(1-2\zeta)\delta]. \end{aligned} \tag{23}$$

Comparing (**??**) to the incentive constraint (**??**) without counting error, the only difference is that $(1-2\zeta)\delta$ replaces δ in the original constraint. Since the magnitude of δ is immaterial in large elections, Proposition **??** continues to hold if we replace the sufficient condition (**??**) by:

$$(1-\alpha)(1-q_l)+\alpha\overline{\theta} < v'_{\mathcal{L}} < v_{\mathcal{C}} < v'_{\mathcal{R}} < (1-\alpha)q_r+\alpha\underline{\theta}.$$

Our second model of counting errors assumes system-wide errors instead of independent mistakes in counting each ballot. For example, such correlated errors may occur when a certain counting protocol (how to deal with hanging chads, etc.) is not properly followed, so that all the votes in the same polling station or even the entire election are miscounted is a specific way. To model these errors, we assume that if the true vote share for candidate $\mathcal{R}$ is v, then upon the initial count the vote share is recorded as

$$v_e = v + u,$$

where u is a random variable with positive and continuous density on the support $[\underline{u}, \overline{u}]$. Upon recounting, all errors are detected so that the election outcome is based on the true vote share v.

The effect of the systematic counting error u is very similar to the effect of aggregate uncertainty θ, except that u only influences the initial vote share but not the final tally. Specifically, if $z(R,\underline{\theta};\kappa^n)+\underline{u} > v_{\mathcal{R}}$, then in state R the pivotal event $v_e = v_{\mathcal{R}}$ dominates the other pivotal events $v = v_{\mathcal{C}}$ and $v_e = v_{\mathcal{L}}$ for sufficiently large n. Proposition **??** continues to hold if we replace the sufficient condition (**??**) by

$$(1-\alpha)(1-q_l)+\alpha\overline{\theta}+\overline{u} < v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}} < (1-\alpha)q_r+\alpha\underline{\theta}+\underline{u}.$$

5.4. Uncertain size of electorate

The analysis presented here can be generalized to the case with an uncertain electorate size if we assume that the number of voters is N, with N being a Poisson random variable with mean n. Myerson (1998; 2000) develops the tools to study such Poisson games.

Recall that from Stirling's approximation to the binomial probability in equation (**??**), the rate at which the pivotal probability that the vote share equals v goes to 0 is given by:

$$\lim_{n\to\infty}\frac{\log g^n(v|S,\theta)}{n} = \log I(v;z(S,\theta;\kappa^n)).$$

In contrast, Myerson (2000) shows that in a Poisson model, the corresponding rate is:

$$\lim_{n\to\infty} \frac{\log g^n(v|S,\theta)}{n} = I(v; z(S,\theta;\kappa^n)) - 1.$$

Since $\log I$ and $I-1$ are positive transformation of one another, given any v, S and κ^n, the θ that maximizes $\log I$ in the model with no population uncertainty also maximizes $I-1$ in the Poisson model. Lemma **??** then implies that if $z(R,\underline{\theta};\kappa^n) > v_{\mathcal{R}}$, then the event $v = v_{\mathcal{R}}$ dominates the events $v = v_{\mathcal{C}}$ and $v = v_{\mathcal{L}}$ in state R. Likewise, if $z(L,\overline{\theta};\kappa^n) < v_{\mathcal{L}}$, then the event $v = v_{\mathcal{L}}$ dominates the events $v = v_{\mathcal{C}}$ and $v = v_{\mathcal{R}}$ in state L. All the results in the current paper remains intact in the Poisson model.

5.5. Possibility of inefficient equilibria

Our result that a sequence of equilibria that achieves asymptotic information efficiency exists does not exclude the possibility that there might be other equilibria whose outcome is far from information inefficiency. Suppose there is an inefficient strategy profile κ^n which is an equilibrium in a standard election with rule $v_{\mathcal{C}}$. By assumption κ^n fails to satisfy

$$z(L,\overline{\theta};\kappa^n) < v_{\mathcal{C}} < z(R,\underline{\theta};\kappa^n) \tag{24}$$

regardless of how large n is. Among these inefficient equilibria, there is a qualitative distinction between the case when (**??**) is violated because for some $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$

$$z(L,\theta;\kappa^n) = v_{\mathcal{C}} = z(R,\theta';\kappa^n) \tag{25}$$

and the other cases. If (**??**) does not hold, any strategy κ^n that fails (**??**) will cease to be an equilibrium in a large election with recount when the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are sufficiently close to $v_{\mathcal{C}}$. This is because in all these equilibria, as the recounting thresholds approach $v_{\mathcal{C}}$, either $g^n(v_{\mathcal{R}}|R)$ or $g^n(v_{\mathcal{L}}|L)$ dominates all other conditional probabilities, implying that it is a best response for informed voters of all preference types and intensities to either vote for $\mathcal{R}$ or vote for $\mathcal{L}$ regardless of their private signal, which cannot be an equilibrium.[12] In contrast, if

[12]There could also be strategy profiles κ^n with $z(R,\overline{\theta};\kappa^n) < v_{\mathcal{L}} < v_{\mathcal{R}} < z(L,\underline{\theta};\kappa^n)$ that are equilibria even as the recounting thresholds become arbitrarily close to $v_{\mathcal{C}}$. However, these equilibria would not survive in a model with two rounds of voting.

the inefficient equilibrium strategy κ^n satisfies (**??**), then it will remain an equilibrium in a large election with recount regardless of the recounting thresholds. For any given standard voting threshold v_C, the set of election environments such that (**??**) has a solution within the strategy space shrinks as the aggregate uncertainty diminishes (i.e., α decreases). Without aggregate uncertainty, (**??**) can only be satisfied non-generically. Thus, choosing recounting thresholds arbitrarily close to v_C guarantees that every sequence of equilibria is asymptotically efficient in a model without aggregate uncertainty, which improves on Battacharya's (2008) result. With aggregate uncertainty, asymptotic efficiency cannot be generically guaranteed. However recounting expands the set of election environments in which an asymptotically efficient sequence of equilibria can be constructed, thus improving on Feddersen and Pesendorfer (1997).

6. Concluding Remarks

This paper is an outgrowth of our earlier papers (Damiano, Li and Suen, 2009; 2010) that use costly delay to improve information aggregation in a two-agent negotiation problem, and to study the design of deadline in negotiations. Here, we introduce multiple pivotal events to resurrect sincere voting in large elections. The key to our equilibrium construction relies on the fact that while the probabilities of different pivotal events are all vanishingly small in large elections, the rate at which they go zero can be ranked. Since the desire to avoid recount cost is preference-independent, and since pivotal events triggering a recount dominate the pivotal event involving a tie between the candidates, we demonstrate how sincere voting or semi-sincere voting can be an equilibrium in large elections with recount, producing aymptotically information efficient outcomes which may otherwise be infeasible in standard elections. The analysis of elections with multiple pivotal events also features in Razin (2003) in the context of signaling policy preference by voters, and in Bouton and Castanheira (2008) and Ahn and Oliveros (2010) in models of multi-candidate and multi-issue voting.

In this paper we have considered the Condorcet jury theorem in large elections. In a jury setting, Austin-Smith and Banks (1996) and Feddersen and Pesendorfer (1998) have shown that the Condorcet jury theorem fails due to strategic voting. In particular, Feddersen and Pesendorfer (1998) show that a unanimous conviction rule in jury decisions may lead to higher probability of false conviction as well as false acquittal than the simple majority rule, and the probability of convicting an innocent defendant may increase with the size of the jury. More relevant to the present paper is a recent literature that asks whether the Condorcet jury theorem continues to hold when acquiring information is costly to individual agents. Mukhopadhaya (2005) shows that in a symmetric mixed strategy equilibrium, as the number of committee members increases, each member chooses to collect information with a smaller probability. He finds examples in which, using the majority rule, a larger committee makes the correct decision with a lower probability than does a smaller one. Koriyama and Szentes (2007) consider a model in which agents choose whether or not to acquire information in the first stage, and then the decision is made according to an ex post efficient rule in the second stage. They show that there is a maximum group size such that in smaller groups each member will choose to collect evidence, and the Condorcet jury theorem fails for larger groups. However, in a model with the quality of information as a continuous choice variable, Martinelli (2006) shows that if the marginal cost of information is near zero for nearly irrelevant information, then there will be effective information aggregation despite the fact that each individual voter will choose to be very poorly informed. In a recent paper, Krishna and Morgan (2010) show that when participation in an election is costly but voluntary, those who choose to participate will vote sincerely even in a standard election. However the fraction of participating voters is vanishingly small in a large election, rendering asymptotic information efficiency difficult to achieve if there is aggregate uncertainty in the model.

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