Chapter 11
GROUP COST OF LIVING INDEXES:
APPROXIMATIONS AND AXIOMATICS*

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1. Introduction

Following Frisch [1936], at least three approaches to constructing a cost of living or consumer price index can be distinguished: (i) the “statistical” approach, (ii) the test or axiomatic approach and (iii) the preference field (cf. Wold [1953; 135]) or economic (cf. Samuelson and Swamy [1974]) approach.

In the statistical approach, price ratios of the form $p_{t+1}^i/p_t^i$ (where $p_t^i$ is the price of commodity $i$ in period $t$, $i = 1, \ldots, N$) are assumed to be stochastically distributed around a common mean. In this approach, the index number problem reduces to finding an appropriate statistical estimator for the common mean. See Frisch [1936] for a review of this approach and Diewert [1981a; 179–180] for a discussion of related “neostatistical” approaches.

In the test or axiomatic approach, the price index is regarded as a function of $4N$ variables, $P(p^0, p^1, x^0, x^1)$, which satisfies certain properties or axioms or tests, where $p_t^i \equiv (p_1^i, p_2^i, \ldots, p_N^i)$ is a vector of period $t$ prices and $x_t^i \equiv (x_1^i, x_2^i, \ldots, x_N^i)$ is a vector of period $t$ commodity consumptions for $t = 0$ and 1. If $N = 1$ so that there is only one commodity, then the price index function is usually defined to be $P(p_0^1, p_1^1, x_0^1, x_1^1) \equiv p_1^1/p_0^1$. The test approach to index number theory was popularized by Irving Fisher [1922] and perfected by Eichhorn [1976] [1978b] and Eichhorn and Voeller [1976] [1983].

The economic approach to the construction of a consumer price index is based on the assumption of utility maximizing behavior on the part of the individual or household during the two periods under consideration. The resulting theoretical price index depends on the (unknown) household preferences. The concept is due to Konüs [1924] and it is known as the Konüs cost of living index. In order to obtain an operational approximation to the theoretical cost of living index, it is usually necessary to make additional assumptions. Thus the economic approach to the construction of a consumer price index may be

broken down into a number of sub approaches: (1) the bounds or nonparametric approach, (2) the exact approach, and (3) the econometric approach.

In the bounds approach, we attempt to find the tightest possible bounds for the theoretical index that can be computed using observable price and quantity data. Of course, we also use the assumption of utility maximizing behavior in all of the economic approaches. Contributors to this approach include Konüs [1924], Samuelson [1947; 159], Malmquist [1953], Pollak [1971a], and Afriat [1972b], [1973d], [1977].

In the econometric approach, we assume the household’s utility function is a specific functional form. Then the observed price and quantity data are used in order to estimate econometrically the parameters of the system of consumer demand equations. Once these parameters have been estimated, the Konüs cost of living index may be calculated. Contributors to this approach include Goldberger and Gamaletos [1970], Christensen and Manser [1975], Braithwaite [1980] and Jorgenson and Slesnick [1983].

In the exact approach, it is again necessary to assume that consumer preferences can be adequately approximated by a certain class of preferences. However, no econometric estimation is necessary in this approach. Theorem 2 below illustrates this approach (while Theorem 1 illustrates the bounds approach). Contributors to the exact approach include Konüs and Byushgens [1926], Afriat [1972b], Samuelson and Swamy [1974] and Diewert [1976a].

This completes our brief review of the alternative approaches to the construction of a consumer price index.

It should be noted that with the exceptions of Pollak [1981], Jorgenson and Slesnick [1983] and Diewert [1983a], the economic approach has centered around the single household case. Thus one of the main purposes of this paper will be to extend the theory to the many households case.

The other main purpose of this paper will be to provide an axiomatic approach to the various concepts of group cost of living indexes. Several years ago, Professor Eichhorn suggested that this axiomatic structure must be possible. Thus it seems appropriate to dedicate this paper to him.

In Section 2, we define the Konüs cost of living index for a single household. We also list some theorems that indicate how approximations to the theoretical index may be computed; i.e., we illustrate the bounds and the exact approaches to the economic theory of index numbers.

In Sections 3 to 5, we define several different theoretical group cost of living indexes and we develop some theorems which indicate how they may be numerically approximated by observable data. We also characterize each theoretical index axiomatically.

Finally, in Section 6 we define a group welfare index, characterize it axiomatically and develop numerical approximations.

2. The Single Household Konüs Cost of Living Index

We assume that the household has preferences over combinations of N goods that may be represented by a utility function F where u = F(x) is the utility level or standard of living that can be attained if the individual consumes the consumption vector x = (x1, x2, . . . , xN)′ ≥ 0N.

We assume that the utility function F defined over {x : x ≥ 0N} satisfies the following conditions I:

(i) continuity,
(ii) increasingness; i.e., if x′′ ≫ x′ ≥ 0N, then F(x′′) > F(x′),
(iii) quasiconcavity; i.e., for each utility level u, the upper level set L(u) ≡ {x : F(x) ≥ u} is convex,
(iv) F(0) = 0, and
(v) F(x) tends to +∞ as the components of x all tend to +∞.

We shall assume that the household maximizes the utility function F(x) subject to a budget constraint of the form p′·x = ∑Nn=1 pnxn ≤ y where p ≫ 0N is a positive vector of commodity (rental) prices and y > 0 is expenditure on the N commodities.1

The household’s utility maximization problem can be decomposed into two stages. In the first stage, the household attempts to minimize the cost of achieving a given utility level, and in the second stage, the maximal utility level that is just consistent with the budget constraint is chosen.

The solution to the first stage problem defines the household’s cost function C: for u ≥ 0, p ≫ 0N,

\[
C(u, p) \equiv \min_x \{p \cdot x : F(x) \geq u, x \geq 0N\}.
\]

Given that F satisfies conditions I, C defined for u ≥ 0, p ≫ 0N will satisfy the following conditions II:

(i) C is continuous,
(ii) C(0, p) = 0 for every p ≫ 0N,
(iii) for every p ≫ 0N, C(u, p) is increasing in u and C(u, p) tends to +∞ as u tends to +∞,
(iv) C(u, p) is positively linearly homogeneous in p for fixed u; i.e., for u ≥ 0, p ≫ 0N and λ > 0, C(u, λp) = λC(u, p),
(v) C(u, p) is concave in p for fixed u,
(vi) C(u, p) is increasing in p for fixed u > 0; i.e., if p′′ ≫ p′ ≫ 0N, u > 0, then C(u, p′′) > C(u, p′), and

1Notation: x′′ denotes the transpose of the column vector x, p′′x = p′ · x = ∑Nn=1 pnxn denotes the inner product of the vectors p and x, x ≥ 0N means each component of the vector x is nonnegative, x ≫ 0N means each component is positive, and x > 0N means x ≥ 0 but x ≠ 0N.
(vii) $C$ is such that the function $F^*(x) \equiv \max_a \{ u : p \cdot x \geq C(u, p) \}$ for every $p \gg 0_N$, $u \geq 0$ is continuous for $x \geq 0_N$.

Moreover, if we are given a cost function $C$ satisfying conditions II, $C$ may be used in order to construct the underlying preference function $F$ which will satisfy conditions I.2

Our interest in $C$ stems from the fact that it may be used to define the Konüs [1924] cost of living index $P_K$: for $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $u > 0$ define

$$P_K(p^0, p^1, u) = C(u, p^0)/C(u, p^0).$$

Thus $P_K$ depends on three variables: (i) $p^0$, a vector of period 0 or base period prices, (ii) $p^1$, a vector of period 1 or current period prices, and (iii) $u$, a number that indexes the reference indifference surface. Thus $P_K(p^0, p^1, u)$ is the minimum cost of achieving the standard of living indexed by $u$ when the household faces period 1 prices $p^1$ relative to the minimum cost of achieving the same standard of living when the household faces period 0 prices $p^0$. If there is only one good, then it can be seen that $P_K(p^0_0, p^1_0, u) = p^1_0/p^0_0$ for all $u > 0$. In this case, there is obviously no index number problem.

In the general case when there is more than one good, the functional form for the cost of living index $P_K$ obviously depends on the functional form for the household’s cost function $C$, which in turn is determined by the form of the household’s preference function $F$. Our fundamental problem is that we do not know what the functional forms for $F$ or $C$, and hence $P_K$, are. Theorems 1 and 2 below indicate how empirically implementable approximations to $P_K$ may be computed. However, first it is useful to introduce the concept of a mechanistic price index formula. This is simply a function $P$ of the observable price and quantity vectors for time periods 0 and 1, $(p^0, p^1, x^0, x^1)$, where $P$ has a known functional form. Two examples of such formulae are the Laspeyres price index $P_L$ defined by

$$P_L(p^0, p^1, x^0, x^1) \equiv p^1 \cdot x^0/p^0 \cdot x^0$$

and the Paasche price index $P_P$ defined by

$$P_P(p^0, p^1, x^0, x^1) \equiv p^1 \cdot x^1/p^0 \cdot x^1.$$  

Irving Fisher [1922] gives hundreds of examples of mechanistic price index formulae. The axiomatic characterization of these indexes may be found in Eichorn [1976] [1978b] and Eichorn and Voeller [1976] [1983]. Note that these indexes are functions of $4N$ arguments whereas the price index that appears in the following Theorem 1 has only $2N + 1$ arguments.

Since the Paasche and Laspeyres indexes are often rather close to each other numerically, Theorem 1 below is a very useful result. However, in order to prove the theorem, we require cost minimizing behavior on the part of the consumer during periods 0 and 1. We shall also assume that we can observe the consumer’s quantity choices $x^0 > 0_N$ and $x^1 > 0_N$ made during periods 0 and 1 in addition to the corresponding price vectors $p^0 \gg 0_N$ and $p^1 \gg 0_N$.

Thus we assume:

$$p^0 \cdot x^0 = C[F(x^0), p^0]; \quad p^1 \cdot x^1 = C[F(x^1), p^1].$$

**Theorem 1.** Konüs [1924: 20–21]: Let the consumer’s utility functions $F$ satisfy conditions I and suppose that the observed data for periods 0 and 1, $(p^0, x^0)$ and $(p^1, x^1)$ respectively, satisfy the cost minimization assumptions (5). Then there exists a reference utility level $u^*$ that lies between the base utility level $u^0 \equiv F(x^0)$ and the period 1 utility level $u^1 \equiv F(x^1)$ such that the consumer’s true cost of living index for this reference utility level, $P_K(p^0, p^1, u^*)$, lies between $P_L \equiv p^1 \cdot x^0/p^0 \cdot x^0$ and $P_P \equiv p^1 \cdot x^1/p^0 \cdot x^1$.

Thus if $P_L$ and $P_P$ are close to each other numerically, an average of $P_L$ and $P_P$, such as Irving Fisher’s [1922] ideal index number $P_F$, will closely approximate the Konüs cost of living index $P_K(p^0, p^1, u^*)$ defined in Theorem 1. $P_F$ is defined as the geometric average of $P_L$ and $P_P$; i.e.,

$$P_F(p^0, p^1, x^0, x^1) \equiv (p^1 \cdot x^0/p^0 \cdot x^0)^{1/2}(p^1 \cdot x^1/p^0 \cdot x^1)^{1/2}.$$  

Theorem 1 above illustrates the bounds approach in the economic theory of index numbers. Theorem 2 below illustrates the exact approach. Before stating Theorem 2, it is first necessary to define the translog cost function $C_T$ by:

$$\ln C_T(u, p) \equiv a_0 + \sum_{i=1}^{N} a_i \ln p_i + (1/2) \sum_{j=1}^{N} a_{ij} \ln p_i \ln p_j + a_{00} \ln u \quad \text{(7)}$$

where the parameters $a_{ij}$ satisfy the following restrictions:

$$\sum_{i=1}^{N} a_i = 1; \quad a_{ij} = a_{ji}; \quad \sum_{j=1}^{N} a_{ij} = 0 \text{ for } i = 1, \ldots, N; \quad \sum_{i=1}^{N} a_{0i} = 0. \quad \text{(8)}$$

It is also necessary to define the base period expenditure shares $s_i^0 \equiv p_i^0 x_i^0/p^0 \cdot x^0$ and the period 1 expenditure shares $s_i^1 \equiv p_i^1 x_i^1/p^1 \cdot x^1$ for $i = 1, \ldots, N$.
Define the translog\(^3\) price index \(P_T\) by

\[
P_T(p^0, p^1, x^0, x^1) = \prod_{i=1}^{N} (p^i_1/p^0_i)^{(x^1_i-x^0_i)/2}.
\]

**Theorem 2.** (Diewert [1976a; 122]): Suppose the household’s cost function \(C\) equals the translog cost function \(C_T\) defined by (7) and suppose the observed data \((p^0, x^0)\) and \((p^1, x^1)\) satisfy the cost minimization assumptions (5) where \(F\) is the utility function dual to \(C\). Define \(u^0 \equiv F(x^0)\), \(u^1 \equiv F(x^1)\) and \(u^* \equiv (u^0 u^1)^{1/2}\). The true cost of living index \(P_K(p^0, p^1, u^*)\) evaluated at the intermediate utility level \(u^*\) may be calculated by evaluating the translog price index \(P_T\); i.e., we have

\[
P_K(p^0, p^1, u^*) \equiv C_T(u^*, u^1)/C_T(u^*, u^0) = P_T(p^0, p^1, x^0, x^1).
\]

We note that the translog function \(C_T\) defined by (7) can provide a second order approximation to an arbitrary twice continuously differentiable cost function, so that Theorem 2 is not restricted to the homothetic case. Thus Theorem 2 provides a very strong economic justification for the use of the translog price index \(P_T\) to approximate the true cost of living \(P_K(p^0, p^1, u^*)\).

Theorem 1 provided a strong justification for the use of the Fisher index \(P_F\) while Theorem 2 justified the use of the translog index \(P_T\). The following theorem tells us that it does not matter very much whether we use \(P_F\) or \(P_T\) in empirical applications: the two index number functions approximate each other to the second order around any equal price and quantity point.

**Theorem 3.** (Diewert [1978b]): The functions \(P_F(p^0, p^1, x^0, x^1)\) and \(P_T(p^0, p^1, x^0, x^1)\) differentially approximate each other to the second order around any point where \(p^0 = p^1 \gg N\) and \(x^0 = x^1 \gg N\); i.e., we have:

\[
P_F(p^0, p^1, x^0, x^0) = P_T(p^0, p^1, x^0, x^0)
\]

\[
\nabla P_F(p^0, p^1, x^0, x^0) = \nabla P_T(p^0, p^1, x^0, x^0)
\]

\[
\nabla^2 P_F(p^0, p^1, x^0, x^0) = \nabla^2 P_T(p^0, p^1, x^0, x^0)
\]

where \(\nabla P\) stands for the 4\(N\) dimensional vector of first order partial derivatives of the function \(P\) and \(\nabla^2 P\) is the \(4N \times 4N\) matrix of second order partial derivatives of \(P\).

This completes our review of the single household theory of the cost of living index. We now turn our attention to the many household case.

\(^3\)This corresponds to the terminology used in Christensen, Cummings and Jorgenson [1980]. Diewert [1981a; 187] called \(P_T\) the Törnqvist price index, but the term translog price index seems to be more descriptive.

### 3. The Plutocratic Cost of Living Index

Assume that there are \(H\) distinct households, where household \(h\) has utility function \(f_h(x)\) satisfying conditions I noted in Section 2 with a dual cost function \(C_h(u_h, p)\).

Let \(p^0 \gg N\) and \(p^1 \gg N\) be the period 0 and 1 price vectors as usual, let \(x^0_h \equiv (x^0_{1h}, x^0_{2h}, \ldots, x^0_{Nh})^T\) be household \(h\)’s observed quantity vector for period \(t\), \(t = 0, 1\) and for \(h = 1, \ldots, H\), let \(u^0_h \equiv F_h(x^0_h)\) and \(u^1_h \equiv F_h(x^1_h)\) be the period 0 and 1 utility levels attained by household \(h\), and define the utility vectors \(u^0 \equiv (u^0_1, \ldots, u^0_H)^T\) and \(u^1 \equiv (u^1_1, \ldots, u^1_H)^T\). Assume cost minimizing behavior for each household during both periods; i.e.,

\[
p^0 \cdot x^0_h = C_h(u^0_h, p^0) = C_h[F_h(x^0_h), p^0] \quad \text{for } h = 1, \ldots, H
\]

\[
p^1 \cdot x^1_h = C_h(u^1_h, p^1) = C_h[F_h(x^1_h), p^1] \quad \text{for } h = 1, \ldots, H.
\]

Pollak [1981; 328] defines what he calls a Scitovsky–Laspeyres cost of living index for a group of households as the ratio of the total expenditure required to enable each household to attain its reference indifference curve at period 1 prices to that required at period 0 prices. This same concept of a group cost of living index was suggested by Prais [1959] in less precise language. Prais referred to his concept as a plutocratic price index. Formally, this Prais–Pollak plutocratic cost of living index may be defined as:

\[
P_{PP}(p^0, p^1, u) \equiv \sum_{h=1}^{H} C_h(u_h, p^1)/\sum_{h=1}^{H} C_h(u_h, p^0)
\]

\[
= \sum_{h=1}^{H} s^h(u, p^0) C_h(u_h, p^1)/C_h(u_h, p^0)
\]

where the household \(h\) share of total expenditure at the vector of reference utility levels \(u \equiv (u_1, \ldots, u_H)\) and prices \(p \equiv (p_1, \ldots, p_N)\) is defined as

\[
s^h(u, p) \equiv C_h(u_h, p)/\sum_{j=1}^{H} C^j(u_j, p) \quad \text{for } h = 1, \ldots, H.
\]

The Laspeyres–Pollak cost of living index is defined as \(P_{PP}(p^0, p^1, u^0)\) while the Paasche–Pollak cost of living index is defined as \(P_{PP}(p^0, p^1, u^1)\).

It is convenient to define aggregate consumption vectors \(\overline{x}\) as follows:

\[
\overline{x}^0 \equiv \sum_{h=1}^{H} x^0_h; \quad \overline{x}^1 \equiv \sum_{h=1}^{H} x^1_h.
\]

Armed with the above definitions, the bounds approach to approximating the plutocratic cost of living index may be illustrated by stating the following two theorems.
Theorem 4. (Diewert [1983a]): Suppose each consumer’s utility function $F^h$ satisfies conditions I and suppose the cost minimization assumption (14) holds. Then

\[
\begin{align*}
(19) & \quad \min_i \{p_i^1/p_i^0\} \leq P_{PP}(p^0, p^1, u^0) \leq p^1 \cdot \varpi^0/p^0 \cdot \varpi^0 \equiv P_L \\
(20) & \quad p_P \equiv p^0 \cdot \varpi^0/p^0 \cdot \varpi^0 \leq P_{PP}(p^0, p^1, u^1) \leq \max_i \{p_i^1/p_i^0\}.
\end{align*}
\]

Note that $P_L$ in (19) is the usual Laspeyres price index involving the aggregate base period consumption vector $\varpi^0$, while $P_P$ in (20) is the usual Paasche price index involving the aggregate period 1 consumption vector $\varpi^1$. The consumer price index constructed by statistical agencies is usually an approximation to $P_L$. All that is required to construct $P_L$ is: the current vector of prices $p^1$, the base price vector $p^0$, and the base period aggregate consumption vector $\varpi^0 \equiv \Sigma_h x_h^0$.

Theorem 5. (Diewert [1983a]): Under the conditions of Theorem 4, there exists a reference utility vector $u^r \equiv (u^r_1, \ldots, u^r_H)$ such that each component $u^r_h$ lies between $u^0_h$ and $u^1_h$, and the Prais–Pollak group cost of living index evaluated at this reference utility vector, $P_{PP}(p^0, p^1, u^r)$, lies between $P_L \equiv p^1 \cdot \varpi^0/p^0 \cdot \varpi^0$ and $P_P \equiv p^1 \cdot \varpi^1/p^0 \cdot \varpi^1$.

Note that the bounds in Theorems 4 and 5 depend only on aggregate price and quantity data. Furthermore, we would expect $P_L$ and $P_P$ to be very close to each other provided that $p^0$ and $p^1$ are not “too” different, and hence $P_{PP}(p^0, p^1, u^r)$ may be closely approximated by either $P_L$ or $P_P$ (or, say, by the Fisher ideal index $P_F \equiv (P_L P_P)^{1/2}$) under these circumstances.

We now turn to an axiomatic characterization of the plutocratic cost of living index.

Theorem 6. Let $P(p^0, p^1, u)$ be a function of $2N + H$ variables defined for $p^0 \gg 0_N, p^1 \gg 0_N, u > 0_H$ satisfying the following properties:

(i) $P(p^0, p^1, u) > 0$ for $p^0 \gg 0_N, p^1 \gg 0_N, u > 0_H$ (positivity);

(ii) $P(p^0, p^1, u) = 1/P(p^1, p^0, u)$ for $p^0 \gg 0_N, p^1 \gg 0_N, u > 0_H$ (time reversal);

(iii) $P(p^0, p^2, u) = P(p^0, p^1, u)P(p^1, p^2, u)$ for $p^0 \gg 0_N, p^1 \gg 0_N, p^2 \gg 0_N, u > 0_H$ (circularity);

(iv) $P(p^*, p, u_1, u_2, \ldots, u_H) = \sum_{k=1}^H u_k P(p^*, p, u_k e_h)$ for some $p^* \gg 0_N$ and for all $p \gg 0_N, u = (u_1, \ldots, u_H) > 0_H$ where $e_h$ is a unit vector with a one in component $h$ and $u_k P(p^*, p, u_k e_h) \leq 0$ if $u_h = 0$ (an additivity across consumers property); and

(v) for the same vector of prices $p^* \gg 0_N$ that occurred in (iv) above, the functions $u_k P(p^*, p, u_k e_h) \equiv C_h^p(u_h, p)$ regarded as functions of $u_h \geq 0$ and $p \gg 0_N$ satisfy conditions II for cost functions for $k = 1, 2, \ldots, H$.

Then $P$ is a plutocratic cost of living index of the form (15) that corresponds to the household preferences that are dual to the functions $C^h$ defined in (v) for $h = 1, \ldots, H$; i.e., $P \equiv P_{PP}$ satisfies (15). Moreover, each $C^h$ satisfies the following money metric scaling of utility property for the reference prices $p^*$:

\[
C^h(u_h, p^*) = u_h \text{ for } u_h > 0, \quad h = 1, \ldots, H.
\]

Conversely, given cost functions $C^h$ satisfying conditions II and the money metric property (21), then $P \equiv P_{PP}$ defined by (15) satisfies properties (i) to (v) listed above in the theorem.

Proof: Let $p^0 \gg 0_N, p^1 \gg 0_N, u \equiv (u_1, u_2, \ldots, u_H) > 0_H$ and let $p^*$ be the vector of prices that occurs in (iv) and (v). Then

\[
P(p^0, p^1, u) = P(p^0, p^*, u)P(p^*, p^1, u) \quad \text{by (iii)}
\]

\[
= P(p^*, p^1, u)/P(p^0, p^*, u) \quad \text{by (ii)}
\]

\[
= \Sigma_h u_h P(p^*, p^1, u_h e_h)/\Sigma_h u_h P(p^*, p^0, u_h e_h) \quad \text{by (iv)}
\]

\[
= \Sigma_h C^h(u_h, p^1)/\Sigma_h C^h(u_h, p^0) \quad \text{by (v)}.
\]

We need to show that the functions $C^h$ defined in property (v) satisfy the money metric property (21). Thus for $h = 1, \ldots, H$ and $u_h > 0$,

\[
C^h(u_h, p^*) = u_h P(p^*, p^*, u_h e_h) = u_h P(p^*, p^*, u_h e_h)P(p^*, p^*, u_h e_h) \quad \text{by (iii)}
\]

\[
= u_h P(p^*, p^*, u_h e_h)/P(p^*, p^*, u_h e_h) \quad \text{by (ii)}
\]

\[
= u_h \quad \text{by (i)}.
\]

The converse part of the theorem is straightforward. QED

Praist [1959] and Muellerbauer [1974] criticized the plutocratic cost of living index because it implicitly gives too much weight to high expenditure households (recall equation (16) above). Thus Praist and Muellerbauer suggested that we take a simple average of the individual household Konüs cost of living indexes, $C^h(u_h, p^1)/C^h(u_h, p^0)$, and this is what they called a democratic price index. We shall find it convenient to consider slightly more general averages, since we may want to weight different households differently, depending on their demographic characteristics (family size, age, etc.). We also find it convenient to consider both arithmetic type averages and geometric type averages. Thus we consider additive type averages in the following section and geometric type averages in the subsection below.

\footnote{The term is due to Samuelson [1974b].}
4. Additive Democratic Cost of Living Indexes

Let $u \equiv (u_1, \ldots, u_H)$ denote a vector of reference utility levels, let $p^0 \gg 0_N$ denote a vector of positive base period prices, let $p^1 \gg 0_N$ denote a vector of current period prices, and let $\alpha > 0_H$ denote a vector of demographic weights such that $\sum_{h=1}^{H} \alpha_h = 1_H \cdot 1 = 1$ where $1_H$ is a vector of ones of dimension $H$. Then we define the additive democratic cost of living $P_D$ as

$$ P_D(p^0, p^1, u, \alpha) \equiv \sum_{h=1}^{H} \alpha_h \frac{C^h(u_h, p^1)}{C^h(u_h, p^0)} $$

Assume cost minimizing behavior for all households during both periods; i.e., assume (14) (and recall the notation immediately above (14)). Recall also Theorem 4 and note that it is valid if $H = 1$. Making repeated use of Theorem 4 applied to individual households, we can obtain the following bounds on the Laspeyres–Democratic index $P_D(p^0, p^1, u^0)$ and the Paasche–Democratic index $P_D(p^0, p^1, u^1, \alpha)$:

$$ \min_i \frac{p^1_i}{p^0_i} \leq P_D(p^0, p^1, u^0, \alpha) \leq \sum_{h=1}^{H} \alpha_h \frac{p^1_h}{p^0_h} $$

$$ \sum_{h=1}^{H} \alpha_h \frac{p^1_h}{p^0_h} \leq P_D(p^0, p^1, u^1, \alpha) \leq \max \frac{p^1_i}{p^0_i} $$

Note that the right hand side of (23) is an arithmetic $\alpha_h$ weighted average of the individual household Laspeyres price indexes, $\sum_h \alpha_h p_L(p^0, p^1, x_h^0, x_h^1)$, which we denote by $P_L^\alpha$. The left hand side of (24) is an arithmetic average of the individual Paasche indexes, $\sum_h \alpha_h p_P(p^0, p^1, x_h^0, x_h^1)$, which we denote by $P_P^\alpha$.

The following theorem provides a group counterpart to Theorem 1.

Theorem 7. Let each consumer’s utility function $F^h$ satisfy conditions I. Suppose that the observed data for periods 0 and 1, $(p^0, x^0_h)$ and $(p^1, x^1_h)$ for $h = 1, \ldots, H$, satisfy the cost minimization assumption (14). Then there exists a reference utility vector $u^* \equiv (u^*_1, \ldots, u^*_H)$ such that each component $u^*_h$ lies between $u^0_h$ and $u^1_h$ and the additive democratic group cost of living evaluated at this reference utility vector $P_D(p^0, p^1, u^*, \alpha)$ lies between $P_L^\alpha$ and $P_P^\alpha$.

Proof: Define $h(\lambda) \equiv P_D[p^0, p^1, (1 - \lambda)u^0 + \lambda u^1]$ for $0 \leq \lambda \leq 1$. Note that $h(0) = P_D(p^0, p^1, u^0, \alpha)$ and $h(1) = P_D(p^0, p^1, u^1, \alpha)$. There are 24 a priori inequality relations that are possible between the four numbers $h(0)$, $h(1)$, $P_L^\alpha$ and $P_P^\alpha$. However, (23) and (24) imply that $h(0) \leq P_L^\alpha$ and $P_P^\alpha \leq h(1)$. This means that there are really only six possible inequalities between the four numbers: (1) $h(0) \leq P_L^\alpha \leq P_P^\alpha \leq h(1)$, (2) $h(0) \leq P_P^\alpha \leq P_L^\alpha \leq h(1)$, (3) $h(0) \leq P_P^\alpha \leq h(1) \leq P_L^\alpha$, (4) $P_L^\alpha \leq h(0) \leq P_P^\alpha \leq h(1)$, (5) $P_P^\alpha \leq h(1) \leq h(0) \leq P_L^\alpha$, and (6) $P_L^\alpha \leq P_P^\alpha \leq h(1) \leq P_L^\alpha$. Since the individual cost functions $C^h(u_h, p^1)$ are continuous in $u_h$, it can be seen that $P_D(p^0, p^1, u, \alpha)$ defined by (22) is continuous in the vector of utility variables $u$. Hence $h(\lambda)$ is a continuous function for $0 \leq \lambda \leq 1$ and assumes all intermediate values between $h(0)$ and $h(1)$. By inspecting cases (1) to (6) above, it can be seen that we can choose $\lambda$ between 0 and 1 (call this number $\lambda^*$) so that $P_L^\alpha \leq h(\lambda^*) \leq P_P^\alpha$ for case (1) or so that $P_P^\alpha \leq h(\lambda^*) \leq P_L^\alpha$ for cases (2) to (6). Now define $u^* \equiv (1 - \lambda^*)u^0 + \lambda^* u^1$ and the proof is complete.

The method of proof used in Theorem 7 (and in Theorems 1 and 4 as well) is due to Končis [1924].

For typical data, we would expect $P_L^\alpha$ and $P_P^\alpha$ to be very close to each other (closer than the individual indexes $P_L^h$ and $P_P^h$ since $P_L^\alpha$ and $P_P^\alpha$ are averages of the individual $P_L^h$ and $P_P^h$), so $P_D(p^0, p^1, u^*, \alpha)$ may be very closely approximated by either $P_L^\alpha$ or $P_P^\alpha$. The practical difficulty with using the democratic price index $P_D(p^0, p^1, u^*, \alpha)$ as a general measure of inflation between periods 0 and 1 is that $P_L^\alpha$ and $P_P^\alpha$ can only be constructed if we have individual household data. The group index defined in the previous section has bounds that could be constructed from aggregate data.

The following theorem provides an axiomatic characterization of the additive democratic price index that is analogous to the axiomatic characterization of the plotonic index that was provided by Theorem 6.

Theorem 8. Let $p^0 \gg 0_N$, $p^1 \gg 0_N$, $p^2 \gg 0_N$, $u \gg 0_H$ and $\alpha > 0_H$ with $1_H \cdot 1 = 1$. Let $P(p^0, p^1, u, \alpha)$ be a function of $2N + 2H$ variables that satisfies the following properties:

(i) $P(p^0, p^1, u, \alpha) > 0$ (positivity);

(ii) for $h = 1, 2, \ldots, H$, $P(p^0, p^1, u_h, e_h) = 1 / P(p^0, p^0, u, e_h)$ where $e_h$ is a unit vector with a one in component $h$ (time reversal);

(iii) $P(p^0, p^1, u_h, e_h) = P(p^0, p^1, u, e_h) P(p^1, p^0, u, e_h)$ for $h = 1, 2, \ldots, H$ (circularity);

(iv) $P(p^0, p^1, u, \alpha) = \sum_{h=1}^{H} \alpha_h P(p^0, p^1, u, e_h)$ (additivity); and

(v) for some positive vector of prices $p^* \gg 0_N$, the function $u_h P(p^*, p, u, e_h)$ $\equiv C^h(u_h, p)$ depends only on $u_h$ and $p \gg 0_N$ and $C^h$ satisfies conditions II for cost functions for $h = 1, 2, \ldots, H$.

Then $P$ is an additive democratic cost of living index of the form (22) that corresponds to the household preferences that are dual to the functions $C^h$ defined in (v) for $h = 1, 2, \ldots, H$. Moreover, each $C^h$ satisfies the money metric scaling of utility property (21) for the reference prices $p^*$. Conversely, given cost functions $C^h$ satisfying conditions II and the money metric property (21)
for \( h = 1, \ldots, H \), then \( P = P_D \) where \( P_D \) defined by (22) satisfies properties (i) to (v) listed above.

Proof: Let \( p^0 \gg 0_N, p^1 \gg 0_N, u \equiv (u_1, u_2, \ldots, u_H) \gg 0_H, \alpha > 0_H \) with \( 1_H \cdot \alpha = 1 \), and let \( p^* \) be the vector of prices that occurs in (v). Then

\[
P(p^0, p^1, u, \alpha) = \sum_{h=1}^{H} \alpha_h P(p^0, p^1, u, e_h) \quad \text{by (iv)}
\]

\[
= \sum_{h=1}^{H} \alpha_h P(p^0, p^*, u, e_h) P(p^*, p^1, u, e_h) \quad \text{by (iii)}
\]

\[
= \sum_{h=1}^{H} \alpha_h u_h P(p^*, p^1, u, e_h) / u_h P(p^*, p^0, u, e_h) \quad \text{by (ii)}
\]

\[
= \sum_{h=1}^{H} \alpha_h C^h(u_h, p^1) / C^h(u_h, p^0) \quad \text{by (v)}
\]

To show that \( C^h \) satisfies the money metric property (21), let \( u_h > 0 \).

Then

\[
C^h(u_h, p^*) \equiv u_h P(p^*, p^*, u, e_h)
= u_h P(p^*, p^*, u, e_h) P(p^*, p^*, u, e_h) \quad \text{by (ii)}
= u_h P(p^*, p^*, u, e_h) P(p^*, p^*, u, e_h) \quad \text{by (ii)}
= u_h \quad \text{by (i)}
\]

The converse part of the theorem is straightforward. \( \Box \)

5. Multiplicative Democratic Cost of Living Indexes

Making the same notational assumptions as at the beginning of Section 4, we may define the multiplicative democratic cost of living index as

\[
P_M(p^0, p^1, u, \alpha) = \prod_{h=1}^{H} \left[ C^h(u_h, p^1) / C^h(u_h, p^0) \right]^{\alpha_h}.
\]

Thus \( P_M \) is a generalized geometric mean of the individual household Konüs cost of living indexes where household \( h \) receives the (demographic) weight \( \alpha_h \). The following theorem provides a group counterpart to Theorem 2.

**Theorem 9.** Suppose household \( h \) has preferences \( F^h \) which are dual to a translog cost function \( C^h \) of the type defined by (7) for \( h = 1, 2, \ldots, H \). Suppose the observed price-quantity data \( (p_t^i, x_t^i, \ldots, x_t^H) \) for period \( t = 0, 1 \) satisfy the cost minimization assumptions (14). Define \( u_t^h \equiv F^h(x_t^h) \) for \( t = 0, 1 \) and \( h = 1, 2, \ldots, H \) and \( u^* \equiv (u_1^*, u_2^*, \ldots, u_H^*) \). Let the demographic weight vector \( \alpha \) satisfy \( \alpha > 0_H \) and \( 1_H \cdot \alpha = 1 \). Then the multiplicative democratic cost of living index \( P_M(p^0, p^1, u^*, \alpha) \) evaluated at the intermediate utility vector \( u^* \) defined above may be calculated by evaluating the right hand side of (26) below:

\[
P_M(p^0, p^1, u^*, \alpha) = \sum_{i=1}^{N} (p^i_1 / p^i_0) \sum_{h=1}^{H} \alpha_h (s^h_0 + s^h_i) / 2
\]

where the period \( t \) expenditure share of commodity \( i \) for consumer \( h \) is defined by

\[
s^h_i \equiv p^i_t x^i_t / p^i / x^t \quad \text{for} \quad t = 0, 1; \quad h = 1, \ldots, H; \quad i = 1, \ldots, N.
\]

Proof: Follows directly from Theorem 2 and definition (25). \( \Box \)

**Corollary.** If \( \alpha_h \equiv 1/H \) for \( h = 1, \ldots, H \), then (26) may be rewritten as

\[
P_M(p^0, p^1, u^*, 1/H) = \prod_{i=1}^{N} (p^i_1 / p^i_0) (s^0_i + s^i_i) / 2
\]

where the average expenditure share for commodity \( i \) during period \( t \) is defined as

\[
s^i_t \equiv \sum_{h=1}^{H} s^h_t / H; \quad t = 0, 1; \quad i = 1, \ldots, N.
\]

It is interesting to contrast Theorem 5 with the Corollary to Theorem 9. If we know total (or average) commodity purchases by households of each commodity, then we may form an accurate approximation to the plutocratic index which appears in the statement of Theorem 5. On the other hand, if we know the average household expenditure shares defined by (28), then we may form an accurate approximation to the (multiplicative) democratic price index \( P_M(p^0, p^1, u^*, 1/H) \).

We conclude this section with an axiomatic characterization of the multiplicative democratic index.

**Theorem 10.** Let \( p^0 \gg 0_N, p^1 \gg 0_N, p^2 \gg 0_N, u \gg 0_H \) and \( \alpha > 0_H \) with \( \alpha \cdot 1_H = 1 \). Let \( P(p^0, p^1, u, \alpha) \) be a function of \( 2N + 2H \) variables that satisfies the following properties:

(i) \( P(p^0, p^1, u, \alpha) > 0 \) (positivity);
(ii) \( P(p^0, p^1, u, \alpha) = 1 / P(p^1, p^0, u, \alpha) \) (time reversal);
(iii) \( P(p^0, p^2, u, \alpha) = P(p^0, p^1, u, \alpha) P(p^1, p^2, u, \alpha) \) (circularity);
(iv) \( P(p^0, p^1, u, \alpha) = \prod_{h=1}^{H} [P(p^0, p^1, u, e_h)]^{\alpha_h} \) (multiplicative property);

and
Then $P$ is a multiplicative democratic cost of living index of the form (25) that corresponds to the household preferences that are dual to the functions $C^h$ defined in (v) for $h = 1, \ldots, H$. Moreover, each $C^h$ satisfies the money metric property (21) for $h = 1, \ldots, H$ for the reference prices $p^*$. Conversely, given cost functions $C^h$ satisfying conditions II and the money metric property (21), then $P \equiv P_M$ where $P_M$ is defined by (25) satisfies properties (i) to (v) listed above.

Proof:

\[
P(p^0, p^1, u, \alpha) = \prod_{h=1}^H \left[ P(p^0, p^1, u, e_h) \right]^{\alpha_h} \quad \text{by (iv)}
\]

\[
= \prod_{h=1}^H \left[ P(p^0, p^*, u, e_h) P(p^*, p^1, u, e_h) \right]^{\alpha_h} \quad \text{by (iii)}
\]

\[
= \prod_{h=1}^H \left[ P(p^*, p^1, u, e_h) / P(p^*, p^0, u, e_h) \right]^{\alpha_h} \quad \text{by (ii)}
\]

\[
= \prod_{h=1}^H u_h P(p^*, p^1, u, e_h) / u_h P(p^*, p^0, u, e_h) \quad \text{by (i)}
\]

\[
= \prod_{h=1}^H \left[ C^h(u_h, p^1) / C^h(u_h, p^0) \right]^{\alpha_h} \quad \text{by (v)}.
\]

The remainder of the proof is straightforward.$\text{QED}$

6. Social Welfare Indexes

For a single household, the Malmquist [1953; 232] quantity index $Q_M$ seems to be an appropriate concept for a utility index. In order to define $Q_M$, we must first define the deflation or distance function $D(u, x)$ that corresponds to the consumer’s utility function $F$. For $u > 0$ and $x > 0_N$, define

\[
D(u, x) = \max_k \{ k : F(x/k) \geq u, \ k > 0 \}.
\]

Thus $D(u, x^1)$ is the deflation factor $k_1$, say, that will just reduce the vector $x^1$ proportionately so that $F(x^1/k_1) = u$. If $F$ satisfies conditions I, then $D$ will satisfy certain regularity conditions (conditions III, say) and a $D$ satisfying these conditions will uniquely characterize $F$.$^5$

For $u > 0$, $x^0 \gg 0_N$, $x^1 \gg 0_N$, define the Malmquist Quantity Index $Q_M$ as

\[
Q_M(x^0, x^1, u) = D(u, x^1) / D(u, x^0).\]

In general, the Malmquist quantity index $Q_M(x^0, x^1, u)$ will depend on the reference indifference surface indexed by $u$.

Turning now to the many consumer case, suppose that household $h$’s preferences may be represented by the deflation function $D^h(u_h, x_h)$ that is dual to the utility function $F^h(x_h)$, where $F^h$ satisfies conditions I as usual.

Define the $N \times H$ matrix of period 0 (1) consumer choices by $X^0(X^1)$; i.e.,

\[
X^0 \equiv (x^0_1, x^0_2, \ldots, x^0_H), \quad X^1 \equiv (x^1_1, x^1_2, \ldots, x^1_H)
\]

where $x_h = (x^t_{1h}, x^t_{2h}, \ldots, x^t_{Nh})^T$ is consumer $h$’s observed consumption vector during period $t$ for $t = 0, 1$ and $h = 1, 2, \ldots, H$.

It is natural to try and aggregate individual welfare changes into a single scalar measure of overall welfare change. A simple index of social welfare change that respects individual preferences is

\[
W(X^0, X^1, u, \beta) \equiv \sum_{h=1}^H \beta_h D^h(u_h, x^1_h) / D^h(u_h, x^0_h)
\]

where $u \equiv (u_1, u_2, \ldots, u_H)$ is a vector of individual reference utility levels and the weights $\beta_h$ satisfy the following restriction:

\[
\sum_{h=1}^H \beta_h = 1.
\]

The reader will note that our indicator of social welfare change defined by (31) is a quantity counterpart to the additive democratic cost of living index defined by (22) (which was an indicator of price change).

Our reason for imposing the restriction (32) is the following: if $x^1_h = x^0_h$ for $h = 1, \ldots, H$ so that there is no change in consumption between periods 0 and 1, then we want our indicator of social welfare change (or our aggregate quantity index) to indicate that there has been no change; i.e., we want $W(X^0, X^0, u, \beta) = 1$. Hence we must have (32).

$^5$The appropriate regularity conditions are listed in Dievert [1982; 560], and references to the literature on duality theorems between $F$ and $D$ may be found there also. Essentially, $D(u, x)$ has the same regularity properties as the cost function $C(u, p)$ where $x$ replaces $p$, except that $D(u, x)$ decreases in $u$ while $C(u, p)$ increases in $u$. 

11. Group Cost of Living Indexes

[300 Essays in Index Number Theory]
In general, the weights \( \beta_h \) do not have to be positive or even nonnegative. However, if the weights are nonnegative, then we obtain the following bounds for \( W \), which are analogous to the bounds in (23) and (24) except that the role of prices and quantities is interchanged:6

\[
\min_{i,h} \{ x_{ih}^1 / x_{ih}^0 \} \leq W(X^0, X^1, u^0, \beta) \leq \sum_{h=1}^H \beta_h p_h \cdot x_h^1 / p_h \cdot x_h^0 = \sum_{h=1}^H \beta_h Q^h_L
\]

and

\[
\sum_{h=1}^H \beta_h Q^h_H \equiv \sum_{h=1}^H \beta_h p_h \cdot x_h^1 / p_h \cdot x_h^0 \leq W(X^0, X^1, u^1, \beta) \leq \max_{i,h} \{ x_{ih}^1 / x_{ih}^0 \}
\]

where \( Q^h_L \) and \( Q^h_H \) are the Laspeyres and Paasche quantity indexes for consumer \( h \), \( u^0 \equiv (u_0^1, \ldots, u_0^H) \), \( u^1 \equiv (u_1^1, \ldots, u_1^H) \) and \( u_h^i \equiv F(x_h^i) \).

Thus the Laspeyres social welfare indicator \( W(X^0, X^1, u^0, \beta) \) is bounded from above by a weighted average of the individual household Laspeyres quantity indexes \( Q^h_L \), and the Paasche social welfare indicator \( W(X^0, X^1, u^1, \beta) \) is bounded from below by a weighted average of the individual household Paasche quantity indexes \( Q^h_H \).

A quantity counterpart to Theorem 7 also holds: if the weights \( \beta_h \) are nonnegative, then there exists a reference utility vector \( u^* = (1 - \lambda^*) u^0 + \lambda^* u^1 \) for some \( \lambda^* \) between 0 and 1 such that the social welfare change index \( W(X^0, X^1, u^*, \beta) \) lies between \( \sum_{h=1}^H \beta_h Q^h_L \) (the upper bound in (33)) and \( \sum_{h=1}^H \beta_h Q^h_P \) (the lower bound in (34)).7

Finally, we have the following theorem which provides an axiomatic characterization for welfare indexes of the form (31).

**Theorem 11.** Let \( X^0 \equiv (x_0^1, \ldots, x_0^H) \), \( X^1 \equiv (x_1^1, \ldots, x_1^H) \) and \( X^2 \equiv (x_2^1, \ldots, x_2^H) \) be \( N \times H \) matrices of positive elements, \( u \gg 0_H \) and \( \beta \gg 0_H \) with \( 1_H \cdot \beta = 1 \). Let \( W(X^0, X^1, u, \beta) \) be a function of \( 2NH + 2H \) variables that satisfies the following properties:

(i) \( W(X^0, X^1, u, \beta) > 0 \) (positivity);
(ii) for \( h = 1, \ldots, H \), \( W(X^0, X^1, u, e_h) = 1/W(X^0, X^1, u, e_h) \) (time reversal);
(iii) for \( h = 1, \ldots, H \), \( W(X^0, X^2, u, e_h) = W(X^0, X^1, u, e_h)W(X^1, X^2, u, e_h) \) (circularity);
(iv) \( W(X^0, X^1, u, \beta) = \sum_{h=1}^H \beta_h W(X^0, X^1, u, e_h) \) (additivity); and
(v) for some positive matrix of reference quantities \( X^* \equiv (x_H^1, \ldots, x_H^1) \), the function \( W(X^*, X^1, u, e_h)/u_h \equiv D^h(u_h, x_h) \) depends only on \( u_h \) and \( x_h \).

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6For a proof of these inequalities, see Diewert [1983a].
7See Diewert [1983a] for a proof.

References for Chapter 11


