A Supplemental Note for "Testing Predictive Ability and Power Robustification."

Kyungchul Song

Department of Economics, University of British Columbia

January 11, 2012

This supplemental note offers a general version of Proposition 1 in "Testing Predictive Ability and Power Robustification (TPA_PR, hereafter)". To define its scope, we need to introduce some notation. Let $(\mathbf{M},\delta)$ be a complete metric space with metric δ on $\mathbf{M} \times \mathbf{M}$, and let $l_\infty(\mathbf{M})$ be the space of bounded functions on $\mathbf{M}$, equipped with the sup norm $||\cdot||_\infty : ||f||_\infty \equiv \sup_{m\in\mathbf{M}}|f(m)|$ for all bounded functions f on $\mathbf{M}$. For any functions $f, g \in l_\infty(\mathbf{M})$, we write $f \geq g$ if $f(m) \geq g(m)$ for all $m \in \mathbf{M}$.

We turn to the hypothesis testing: for a given map $d \in l_\infty(\mathbf{M})$,

$$\begin{aligned} H_0 &: \quad d(m) \leq 0 \text{ for all } m \in \mathbf{M} \text{ against} \\ H_1 &: \quad d(m) > 0 \text{ for some } m \in \mathbf{M}. \end{aligned}$$

Let $\hat{d}(m)$ be an estimator of $d(m)$ such that as $n \to \infty$,

$$\sqrt{n}\{\hat{d} - d\} \Longrightarrow Z \text{ in } l_\infty(\mathbf{M}), \tag{1}$$

where Z is a separable Gaussian process in $l_\infty(\mathbf{M})$ with respect to the distance δ on $\mathbf{M}$ and $\Longrightarrow$ denotes weak convergence in the sense of Hoffman-Jorgensen (e.g. van der Vaart and Wellner (1996)). (For separability of a stochastic process, see the footnote on p. 98 of van der Vaart and

Wellner (1996).) Using $\hat{d}$, we consider the following test statistic:

$$T^K \equiv \sup_{m \in \mathbf{M}} \hat{d}(m).$$

We introduce Pitman local alternatives. For given continuous $d_0 \in l_\infty(\mathbf{M})$ such that $d_0(\cdot) \leq 0$, we take $\mathbf{M}_0$ to be the *zero-set* $\mathbf{M}_0 \equiv \{m \in \mathbf{M} : d_0(m) = 0\}$. Let $A_0 \equiv \{a \in l_\infty(\mathbf{M}) : \sup_{m \in \mathbf{M}_0} a(m) > 0\}$ and $A_{00} \subset A_0$ be such that for all $a \in A_0$ there exists $a_1 \in A_{00}$ satisfying $a_1 \leq a$. For given such $d_0 \in l_\infty(\mathbf{M})$ and $a \in A_{00}$, Pitman local alternatives in the direction a are defined as a sequence of probabilities under which

$$d(m) = d_0(m) + a(m)/\sqrt{n}. \tag{2}$$

To analyze the local power properties of the test T^K in TPA_PR, it suffices to consider local alternatives under which the one-side sup test has a nondegenerate limiting distribution, for which it is necessary that the zero-set $\mathbf{M}_0$ is nonempty. Given the distance δ on $\mathbf{M}$, we define a map $\Psi_\delta(\cdot; \mathbf{M}_0) : [0, \infty) \to [0, \infty)$ by

$$\Psi_\delta(\nu; \mathbf{M}_0) \equiv \int_0^\nu \sqrt{\log D(\varepsilon, \mathbf{M}_0, \delta)} d\varepsilon,$$

where $D(\varepsilon, \mathbf{M}_0, \delta)$ is a packing number, i.e., the maximum number of ε-separated (with respect to δ) points in $\mathbf{M}_0$. Proposition A1 below is a generalized version of Proposition 1 in TPA_PR.

Proposition A1 : *Let Z be a separable centered Gaussian process having a continuous sample path in $l_\infty(\mathbf{M})$. Assume that $\lim_{\nu \downarrow 0} \Psi_\delta(\nu; \mathbf{M}_0) = 0$. Then for any $\varepsilon > 0$, $\alpha \in (0, 1]$, and $c_\alpha > 0$ with $\lim_{n \to \infty} P\left\{T^K > c_\alpha\right\} \leq \alpha$ under H_0, there exists $a \subset A_{00}$ such that*

$$\lim_{n \to \infty} P_a \left\{T^K > c_\alpha\right\} \leq \alpha - \Delta + \varepsilon,$$

where P_a *denotes the sequence of probabilities under* (2) *and* $\Delta \equiv P\{\sup_{m\in\mathbf{M}_0} Z(m) > c_\alpha\} - \inf_{m\in\mathbf{M}_0} P\{Z(m) > c_\alpha\}$.

Proof: For any subset $\mathbf{M}_1 \subset \mathbf{M}$ and any function f on $\mathbf{M}$, define the one-sided sup functional $\Gamma(f;\mathbf{M}_1) = \sup_{m\in\mathbf{M}_1} f(m)$. Take c_α such that $P\{\Gamma(Z;\mathbf{M}_0) > c_\alpha\} \leq \alpha$. Fix $\varepsilon > 0$ and $m_0 \in \mathbf{M}_0$. Take small $\eta > 0$ and large $b \in (0,\infty)$ such that

$$\begin{aligned} P\{Z(m_0) \leq c_\alpha - 3\eta/2\} &\geq P\{Z(m_0) \leq c_\alpha\} - \varepsilon/2 \text{ and} \\ P\{\Gamma(Z - Z(m_0);\mathbf{M}_0) \geq b + \eta/2\} &\leq \varepsilon/6. \end{aligned} \tag{3}$$

Define $J_0^x \equiv \{m \in \mathbf{M}_0 : \delta(m, m_0) \leq x\}$, $x \in [0,\infty)$, and $J_1^x = \mathbf{M}_0 \backslash J_0^x$. Since Z is a separable Gaussian process, we find $x_2 > 0$ such that for all $x \in (0, x_2]$,

$$P\{|\Gamma(Z - Z(m_0); J_0^x)| > \eta/2\} \leq \frac{2}{\eta}\mathbf{E}\left[|\Gamma(Z - Z(m_0); J_0^x)|\right] < \frac{\varepsilon}{6}. \tag{4}$$

(See e.g. Corollary 2.2.8 of van der Vaart and Wellner (1996).) Fix $x \in (0, x_2]$ and define $D_x(m) \equiv \eta 1\{m \in J_0^x\} - b1\{m \in J_1^x\}$ and the event

$$\tau_x(Z) \equiv \{m \in \mathbf{M}_0 : \Gamma(Z + D_x;\mathbf{M}_0) = (Z + D_x)(m)\}. \tag{5}$$

Note that $\mathbf{M}_0$, being a closed subset of $\mathbf{M}$, is complete, and also by the assumption of the proposition, it is totally bounded with respect to δ. Hence $\mathbf{M}_0$ is compact in $\mathbf{M}$. Since $Z + D_x$ is upper semicontinuous, $\tau_x(Z)$ is not empty, meaning that we have either $\tau_x(Z)$ meets with J_0^x or meets with J_1^x but not both, because J_1^x and J_0^x are disjoint. Hence

$$\begin{aligned} P\left\{\tau_x(Z) \cap J_0^x = \varnothing\right\} &= P\left\{\Gamma(Z + D_x; J_1^x) \geq \Gamma(Z + D_x; J_0^x)\right\} \\ &= P\left\{\Gamma(Z; J_1^x) - b \geq \Gamma(Z; J_0^x) + \eta\right\} \\ &\leq P\left\{\Gamma(Z; \mathbf{M}_0) - b \geq Z(m_0) + \eta/2\right\} + \varepsilon/6 \\ &= P\left\{\Gamma(Z - Z(m_0); \mathbf{M}_0) \geq b + \eta/2\right\} + \varepsilon/6 \\ &\leq \varepsilon/3. \end{aligned} \tag{6}$$

The second inequality is due to (4) and due to the fact that $\Gamma(Z; J_1^x) \leq \Gamma(Z; \mathbf{M}_0)$. The third inequality is due to the second bound in (3). Now, observe that for any $a \leq D_x$,

$$\begin{aligned} P\left\{\Gamma\left(Z + a; \mathbf{M}_0\right) \leq c_\alpha\right\} &\geq P\left\{\Gamma\left(Z + D_x; \mathbf{M}_0\right) \leq c_\alpha, \tau_x(Z) \cap J_0^x = \varnothing\right\} \\ &\quad + P\left\{\Gamma\left(Z + D_x; \mathbf{M}_0\right) \leq c_\alpha, \tau_x(Z) \cap J_0^x \neq \varnothing\right\} \\ &\geq P\left\{\Gamma\left(Z + D_x; J_0^x\right) \leq c_\alpha, \tau_x(Z) \cap J_0^x \neq \varnothing\right\} \\ &= P\left\{\Gamma\left(Z; J_0^x\right) \leq c_\alpha - \eta, \tau_x(Z) \cap J_0^x \neq \varnothing\right\} \\ &\geq P\left\{Z(m_0) \leq c_\alpha - 3\eta/2, \tau_x(Z) \cap J_0^x \neq \varnothing\right\} - \varepsilon/6 \\ &\geq P\left\{Z(m_0) \leq c_\alpha - 3\eta/2\right\} - \varepsilon/2 \\ &\geq P\left\{Z(m_0) \leq c_\alpha\right\} - \varepsilon. \end{aligned}$$

The third inequality is due to the the bound in (4), the fourth, the bound in (6) and the last, the first bound in (3). Since the choice of m_0 was arbitrary, we conclude that

$$\begin{aligned}
P\left\{\Gamma\left(Z+a;\mathbf{M}_0\right)\le c_\alpha\right\} &\ge \sup_{m\in\mathbf{M}_0} P\left\{Z(m)\le c_\alpha\right\}-\varepsilon, \text{ or} \\
P\left\{\Gamma\left(Z+a;\mathbf{M}_0\right)> c_\alpha\right\} &\le 1-\sup_{m\in\mathbf{M}_0} P\left\{Z(m)\le c_\alpha\right\}+\varepsilon \\
&= \inf_{m\in\mathbf{M}_0} P\left\{Z(m)> c_\alpha\right\}+\varepsilon \\
&= P\left\{\Gamma\left(Z;\mathbf{M}_0\right)> c_\alpha\right\}-\Delta+\varepsilon.
\end{aligned}$$

By noting that $\lim_{n\to\infty} P_a\left\{T^K > c_\alpha\right\} = P\left\{\Gamma\left(Z+a;\mathbf{M}_0\right)\le c_\alpha\right\}$ (under the local alternatives) and $\lim_{n\to\infty} P\left\{T^K > c_\alpha\right\} = P\left\{\Gamma\left(Z;\mathbf{M}_0\right)\le c_\alpha\right\}$ (under the null hypothesis), we obtain the wanted result. ■

Remark 1 : Note that when there exists $m_0 \in \mathbf{M}$ such that $P\{Z(m_0) = 0\} = 1$ (such as when $\mathbf{M} = [0,1]$ and Z is a centered Brownian motion or Brownian bridge starting at 0), the rejection probability in Proposition A1 can be reduced to an arbitrarily small number. Therefore, the maximin power over $a \in A_{00}$ is zero in this case.

2 : A similar argument can be used to show the asymptotic biasedness for the one-sided intergral-type test: $T = \int \max\{\sqrt{n}\hat{d}(m_u), 0\}^p d\mu(u)$ where $\mathbf{M} = \{m_u : u \in \mathbf{R}^d\}$ and μ is a measure on $\mathbf{R}^d$. The latter type of tests were used by Linton, Song, and Whang (2010).

3 : The condition $\lim_{\nu\downarrow 0} \Psi_\delta(\nu;\mathbf{M}_0) = 0$ requires that the class of functions $\mathbf{M}_0$ not be too large, and is satisfied by many classes of functions that are considered in the empirical process theory. For example, if $\mathbf{M}$ is a Glivenko-Cantelli class with respect to δ equal to an $L_2(P)$-norm, the condition is satisfied.

4 : Theorem 1 does not require that finite dimensional distributions of the Gaussian process Z have a positive definite covariance matrix. This has a significance consequence in testing predictive ability. See TPA_PR for details.

References

[1] Linton, O., K. Song, and Y-J. Whang (2010), "An improved bootstrap test of stochastic dominance," *Journal of Econometrics* 154, 186-202.

[2] van der Vaart, A. W. and J. A. Wellner (1996), *Weak Convergence and Empirical Processes*, Springer-Verlag.