Volume Title: Annals of Economic and Social Measurement, Volume 4, number 3

Volume Author/Editor: NBER

Volume Publisher:

Volume URL: http://www.nber.org/books/aesm75-3

Publication Date: July 1975

Chapter Title: Recursive Estimation Algorithms for Economic Research

Chapter Author: Craig Riddell

Chapter URL: http://www.nber.org/chapters/c10407

Chapter pages in book: (p. 397 - 406)

*Annals of Economic and Social Measurement*, 4/3, 1975

# RECURSIVE ESTIMATION ALGORITHMS FOR ECONOMIC RESEARCH

BY WM. CRAIG RIDDELL*

*Recursive estimation algorithms can be used to update estimates to account for new data. Such algorithms can also be used to approve the robustness of the model within the sample period. Both single and multiple equation systems are examined.*

## I. INTRODUCTION

"On-line", "recursive" and "sequential" estimation are terms used to describe procedures whereby parameter estimates for the current period[1] are calculated from current data and previous estimates only. These methods of updating estimates as additional data arrives have considerable computational advantages over the standard practice of re-estimation using the entire (larger) data set. In addition to computational savings, storage requirements can be reduced as past data need not be retained. This factor is crucial in real-time estimation; for example, orbital determination for spacecraft (see, e.g. [3]). Here, storage limitations of on board computers combined with frequently arriving data and the need for estimates that fully reflect the most recently collected observations combine to make recursive estimation necessary.

While there may be less motivation for the use of these methods in economic and social research, their potential should not be ignored. In addition to providing an inexpensive method of updating estimates to account for new data, these algorithms can also be useful in appraising the robustness of the model within the sample period. There is little extra cost involved in obtaining estimates for $t = T^*, T^* + 1, \ldots T$ over that for $t = T$ alone. The results may point to a need for re-specification of the (assumed fixed parameter) model or adoption of a time-varying parameter model. This type of "stability analysis" has been used, for example, by Fair ([12], Ch. 12) who re-estimates each equation of his model for samples of 33 to 50 observations inclusive and by Goldfeld [14] who re-estimates a quarterly model at annual intervals.[2]

Section II of the paper provides a survey of algorithms for single equation models. Extensions to multiple equation systems are considered in Section III.

## II. SINGLE EQUATION MODELS

While Carl Gauss [13] gave the formulae applicable to least squares when an additional observation is received, recursive algorithms date from the papers of Sherman and Morrison [24], [25], Plackett [22], and Bartlett [5]. These authors derived matrix identities useful in calculating the inverse of a matrix $B$ from the

[1] While the discussion relates to time series data, all results are applicable to cross-section data.

[2] As will be seen, recursive algorithms are not yet available for the error specifications employed by these authors.

* Department of Economics, Queen's University, Kingston, Canada. Thanks to J. C. R. Rowley, Economic Council of Canada, for suggesting this topic: and Charles Beach, Queen's University, and a referee for comments on an earlier draft. Errors and ommissions are the responsibility of the author. Financial support of the Canada Council is gratefully acknowledged.

inverse of $A$ when $B$ results from a specified change in $A$. These results are collectively known as the "matrix inversion lemma" (Sage and Melsa [23], Appendix A). One such result is

$$\text{M1.}\quad B^{-1} = (A + U\Sigma V')^{-1} = A^{-1} - A^{-1}U(\Sigma^{-1} + V'A^{-1}U)^{-1}V'A^{-1}$$

where $A$ and $\Sigma$ are square, $U$ and $V$ are rectangular and the necessary inverses are assumed to exist. Computational savings result when the order of $(\Sigma^{-1} + V'A^{-1}U)$ is considerably less than that of $B$. For example, the special case

$$\text{M2.}\quad B^{-1} = (A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$$

where $u$ and $v$ are column vectors requires only the inversion of a scalar.

(a) *The Classical Linear Model*

Consider the familiar model

$$y(T) = X_T\beta + u(T) \tag{1}$$

where

$$y(T) = (y_1, \dots y_T)', \qquad \text{a } T \text{ by 1 vector}$$

$$X_T = (x_1, \dots x_T)', \qquad \text{a } T \text{ by } K \text{ matrix.}$$

Then under the assumptions

A1. $E[u(T)] = 0$

A2. $\operatorname{Cov}[u(T)] = \sigma^2 I_T$

A3. $X_T$ is a non-stochastic matrix of rank $K$

the BLUE is

$$\hat{\beta}(T) = (X'_T X_T)^{-1} X'_T y(T). \tag{2}$$

At time $(T + 1)$ the additional observations $(y_{T+1}, x_{T+1})$ are received. Applying M2 gives the updated covariance matrix of the estimates

$$\begin{aligned}(X'_{T+1}X_{T+1})^{-1} &= (X'_T X_T + x'_{T+1}x_{T+1})^{-1} \\ &= (X'_T X_T)^{-1} - \frac{(X'_T X_T)^{-1} x'_{T+1} x_{T+1} (X'_T X_T)^{-1}}{d}\end{aligned} \tag{3}$$

where

$$d = 1 + x_{T+1}(X'_T X_T)^{-1} x'_{T+1}. \tag{4}$$

Using (3) the OLS recursive algorithm can be obtained:

$$\hat{\beta}(T + 1) = \hat{\beta}(T) + k_{T+1}[y_{T+1} - x_{T+1}\hat{\beta}(T)] \tag{5}$$

where

$$k_{T+1} = \frac{(X'_T X_T)^{-1} x'_{T+1}}{d}. \tag{6}$$

The updating formula is in "differential correction" form, the correction term being proportional to the prediction error. The vector of proportionality $k_{T+1}$ is often termed the "smoothing vector" in the engineering and control literature.

The residual sum of squares can be updated (Duncan and Jones [11]):

$$\hat{S}(T+1) = \hat{S}(T) + \frac{[y_{T+1} - x_{T+1}\hat{\beta}(T)]^2}{d} \tag{7}$$

The OLS recursive estimation procedure is summarized below. At the beginning of each step, the following are available in storage:

$$\begin{array}{ccc} \hat{\beta}(T) & (X_T'X_T)^{-1} & \hat{S}(T) \\ (K \text{ by } 1) & (K \text{ by } K) & (\text{Scalar}). \end{array}$$

Upon receipt of $x_{T+1}$ (1 by $K$) and $y_{T+1}$ (scalar), $d$ is calculated from (4), $(X_{T+1}' \times X_{T+1})^{-1}$ from (3), $\hat{\beta}(T+1)$ from (5), and $\hat{S}(T+1)$ from (7). If the additional assumption of normally distributed errors is made, both $t$ and $F$ statistics can be calculated at each stage. The coefficient of multiple determination can be obtained by retaining the scalars $y_*'y_*(T)$ and $\bar{y}_T$ in storage and using

$$y_*'y_*(T+1) = y_*'y_*(T) + \left(\frac{T}{T+1}\right)(y_{T+1} - \bar{y}_T)^2 \tag{8}$$

where

$$\bar{y}_T = \frac{e'}{T}y(T)$$

$$y_*(T) = y(T) - e\bar{y}_T$$

and $e$ is a ($T$ by 1) vector of unit elements. Then

$$R_{T+1}^2 = 1 - \frac{\hat{S}(T+1)}{y_*'y_*(T+1)}. \tag{9}$$

The formulae for an additional group of observations (say $n$) are derived in a similar manner. Using the notation

$$y(T+n) = \begin{bmatrix} y(T) \\ y(n) \end{bmatrix} \qquad X_{T+n} = \begin{bmatrix} X_T \\ X_n \end{bmatrix}$$

then

$$\hat{\beta}(T+n) = \hat{\beta}(T) + K_{T+n}[y(n) - X_n\hat{\beta}(T)] \tag{10}$$

$$K_{T+n} = (X_T'X_T)^{-1}X_n'(I_n + X_n(X_T'X_T)^{-1}X_n')^{-1}. \tag{11}$$

The obvious disadvantage of this procedure is the need for inversion of a matrix of order $n$ at each stage. Generally it will be simpler to add observations one at a time even when a listing of all the revised estimates is not desired.

The OLS recursive algorithms presented above assume that the normal equations

$$X'X\hat{\beta} = X'y$$

are solved by forming the inverse of $X'X$. This matrix, however, may be ill-conditioned and therefore overly influenced by roundoff errors. Alternative methods recommended by numerical analysts utilize decompositions of either $X$ or $X'X$. Details of these, together with updating procedures are available in Golub [15] and Chambers [10].

In some circumstances, A3 is not valid and is therefore relaxed to A3'. $X_T$ is a non-stochastic matrix of rank $p \leq K$. Under this more general assumption, the least squares estimate is

(12) $$\hat{\beta}(T) = X_T^+ y(T)$$

where $A^+$ denotes the (unique) Moore–Penrose inverse of $A$. That is,

$$AA^+A = A$$

$$A^+AA^+ = A^+$$

$$A^+A \text{ and } AA^+ \text{ are symmetric.}$$

The updating formulae are obtained by using the following theorem (Greville [16]):
Let

$$X_{T+1} = \begin{bmatrix} X_T \\ x_{T+1} \end{bmatrix}.$$

Then

$$\text{M3.} \quad X_{T+1}^+ = [(I - k_{T+1}x_{T+1})X_T^+ . k_{T+1}]$$

where

(13a) $$k_{T+1} = \frac{(I_K - X_T^+ X_T)x'_{T+1}}{x_{T+1}(I_K - X_T^+ X_T)x'_{T+1}} \qquad \text{if } (I_K - X_T^+ X_T)x'_{T+1} \neq 0$$

(13b) $$k_{T+1} = \frac{X_T^+ X_T^{+\prime} x'_{T+1}}{1 + x_{T+1} X_T^+ X_T^{+\prime} x'_{T+1}} \qquad \text{otherwise.}$$

Combining (12) and M3 gives the familiar formula (5) where $k_{T+1}$ is now defined by either (13a) of (13b). When $p = K$, (13b) reduces to (6).

Two extensions of the OLS updating procedure deserve mention. Jones [18] provides the appropriate formulae when only a subset of the regression coefficients are of interest. Albert and Sittler [2] obtain the recursive algorithm for estimation subject to a set of linear constraints.

(b) *The Generalized Linear Model*

In many circumstances, A2 is considered unduly restrictive and is replaced by

$$\text{A2}'. \quad \mathrm{Cov}\,[u(T)] = \sigma^2 V_T$$

where $V_T$ is a known positive definite matrix. $V_T$ is assumed below to be of full rank. The BLUE is the Aitken estimator

(14) $$\hat{\beta}(T) = (X'_T V_T^{-1} X_T)^{-1} X'_T V_T^{-1} y(T).$$

The general recursive form of (14) has not been derived; however, various authors have obtained recursive algorithms by suitably restricting the covariance matrix $V_T$. Albert and Sittler [2] consider the case of heteroskedastic errors (i.e. $V_T = dg(\sigma_1^2, \ldots \sigma_T^2)$) and Blum [7], [8] derives the algorithm appropriate when the error term obeys a $p$th order autoregressive process. Since it is desirable to have an algorithm useful for any choice of the covariance matrix, this is obtained below. Various special cases can then be examined.

Since $V_T$ is positive definite, we can obtain a non-singular matrix $H_T$ such that

(15) $$H_T V_T H'_T = I_T$$

where $H_T$ is lower triangular and can be generated recursively with respect to the index $T$. That is,

(16) $$H_{T+1} = \begin{bmatrix} H_T & 0 \\ h_{T+1} & h \end{bmatrix}.$$

Now (14) can be written as

(17) $$\hat{\beta}(T) = (G'_T G_T)^{-1} G'_T z(T)$$

where

$$G_T = H_T X_T$$

$$z(t) = H_T y(T).$$

Thus

(18) $$G_{T+1} = \begin{bmatrix} G_T \\ h_{T+1} X_T + h x_{T+1} \end{bmatrix} \equiv \begin{bmatrix} G_T \\ g_{T+1} \end{bmatrix}$$

and

$$z(T+1) = \begin{bmatrix} z(T) \\ h_{T+1} y(T) + h y_{T+1} \end{bmatrix} \equiv \begin{bmatrix} z(T) \\ z_{T+1} \end{bmatrix}.$$

The covariance matrix of the estimates is updated by applying M2

(19) $$(G'_{T+1} G_{T+1})^{-1} = (G'_T G_T)^{-1} - \frac{(G'_T G_T)^{-1} g'_{T+1} g_{T+1} (G'_T G_T)^{-1}}{1 + g_{T+1} (G'_T G_T)^{-1} g'_{T+1}}$$

and the generalized least squares recursive algorithm is

(20) $$\hat{\beta}(T+1) = \hat{\beta}(T) + \frac{(G'_T G_T)^{-1} g'_{T+1} [z_{T+1} - g_{T+1} \hat{\beta}(T)]}{1 + g_{T+1} (G'_T G_T)^{-1} g'_{T+1}}.$$

While implementation of (19) and (20) is computationally efficient relative to re-estimation, storage requirements are greater than for OLS as calculation of $g_{T+1}$ and $z_{T+1}$ requires past data. For the case of heteroskedastic errors this difficulty does not arise as $h_{T+1}$ is the null vector and $h = 1/\sigma_{T+1}$. Thus

(21) $$\hat{\beta}(T+1) = \hat{\beta}(T) + \frac{(G'_T G_T)^{-1} x'_{T+1} [y_{T+1} - x_{T+1} \hat{\beta}(T)]}{\sigma_{T+1}^2 + x_{T+1} (G'_T G_T)^{-1} x'_{T+1}}$$

when the error term obeys a $p$th order autoregressive process, the last $p$ observations are required in storage. For illustrative purposes, the case $p = 1$ is considered below. Writing $u_t = \alpha u_{t-1} + \varepsilon_t$ in matrix form gives $u(T) = R_T \varepsilon(T)$, where

$$R_T^{-1} = H_T = \frac{1}{(1-\alpha^2)^{1/2}} \begin{bmatrix} (1-\alpha^2)^{1/2} & & & & 0 \\ -\alpha & 1 & & & \\ & -\alpha & 1 & & \\ 0 & & \ddots & \ddots & \\ & & & -\alpha & 1 \end{bmatrix}$$

Therefore

$$g_{T+1} = x_{T+1} - \alpha x_T$$

and

$$z_{T+1} = y_{T+1} - \alpha y_T$$

are used in (19) and (20).

The procedure for moving average errors is derived in a similar fashion. However, computational requirements will usually be greater than those for auto-regressive errors. This follows from the fact that the lower triangular portion of $H_T$ will, in general, not contain zero elements. Thus $X_T$ and $y(T)$ may be required in storage in order to calculate $g_{T+1}$ and $z_{T+1}$. This burden can be reduced considerably for those cases in which a simple pattern links the vectors $h_T$ and $h_{T+1}$. In these circumstances, $h_{T+1}X_T$ can be obtained from $h_T X_T$ which has already been calculated.

For a large number of economic applications, the covariance matrix $V_T$ is not known and must therefore be estimated. Recursive estimation procedures are not yet available for these cases.

## III. Multiple Equation Models

In this section, the feasibility of recursively estimating parameters of a system of equations is examined. For previous literature, see Odell and Lewis [21] who consider a specification encountered infrequently in economics. Two familiar specifications are examined here.

### (a) *"Seemingly Unrelated" Regressions*

Consider the $m$ equation system

(22) $$Y = X\beta + U$$

where $Y$ is $mT$ by 1, $X = dg(X_1, \ldots X_m)$ is $mT$ by $K$.

It is assumed that

$$E(U) = 0$$

$$\mathrm{Cov}(U) = \Sigma \otimes I_T$$

$X$ is a non-stochastic matrix of rank $K$. Denoting the augmented data at time $(T + 1)$ by

$$\begin{bmatrix} Y \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X \\ x \end{bmatrix}$$

where $y$ is $m$ by 1, $x$ is $m$ by $K$, block diagonal, then the updating formulae can be obtained as:

$$(G'_{T+1}G_{T+1})^{-1} = (G'_TG_T)^{-1} - (G'_TG_T)^{-1}x'D^{-1}x(G'_TG_T)^{-1} \tag{23}$$

and

$$\hat{\beta}(T + 1) = \hat{\beta}(T) + (G'_TG_T)^{-1}x'D^{-1}[y - x\hat{\beta}(T)] \tag{24}$$

where

$$G_T = (L' \otimes I_T)X$$

$$L'\Sigma L = I_m$$

and

$$D = \Sigma + x(G'_TG_T)^{-1}x'.$$

The algorithm is computationally attractive, although it does involve inversion of the matrix $D$ at each stage. For most applications, however, the contemporaneous covariance matrix $\Sigma$ is not known and is estimated by applying OLS to each equation *seriatum* and using the OLS residuals to form

$$\hat{\Sigma} = \frac{1}{T}(\hat{u}_i(T)'\hat{u}_j(T)).$$

While the diagonal elements of $\hat{\Sigma}$ can be updated using (7), computation of the off-diagonal elements requires all past data. This limits the applicability of (24).

(b) *Two Stage Least Squares* (TSLS)

The first of $m$ structural equations is written as

$$y_1(T) = Y_1(T)\beta_1 + X_2(T)\gamma_1 + u_1(T) \tag{25}$$

and the entire system as

$$Y(T) = Y(T)B^* + X(T)C^* + U(T) \tag{26}$$

where $Y_1$, $X_1$, $Y$, $X$ have $m_1$, $G_1$, $m$, $G$ rows respectively. The discussion of the TSLS estimator is often cast in intuitively appealing terms by considering the first stage as that of "purging" the included endogenous variables $Y_1(T)$ of their stochastic component, the second stage consisting of OLS estimation with the adjusted data. In this terminology, the derivation of the TSLS recursive algorithm is complicated by the fact that not only do the new observations on the included endogenous variables require "purging" but also all of the previous observations must be "repurged" to take account of new data.

The following notation will be employed:

$$y_1(T+1) = \begin{pmatrix} y_1(T) \\ y \end{pmatrix}, \; Y_1(T+1) = \begin{pmatrix} Y_1 \\ y_1 \end{pmatrix}, \; X(T+1) = \begin{pmatrix} X \\ x \end{pmatrix}, \; X_1(T+1) = \begin{pmatrix} X_1 \\ x_1 \end{pmatrix}.$$

The TSLS estimator at $(T+1)$ is given by

$$\hat{\delta}_1(T+1) = \begin{pmatrix} \hat{\beta}_1(T+1) \\ \hat{\gamma}_1(T+1) \end{pmatrix} = D_{T+1}^{-1} F_{T+1} \tag{27}$$

where

$$D_{T+1} = \begin{bmatrix} (Y_1'X + y_1'x)(X'X + x'x)^{-1}(X'Y_1 + x'y_1) & Y_1'X_1 + y_1'x_1 \\ X_1'Y_1 + x_1'y_1 & X_1'X_1 + x_1'x_1 \end{bmatrix}$$

$$F_{T+1} = \begin{bmatrix} (Y_1'X + y_1'x)(X'X + x'x)^{-1}(X'y_1(T) + x'y) \\ X_1'y_1(T) + x_1'y \end{bmatrix}.$$

Using M2 it can be shown that

$$D_{T+1} = D_T + z'z - \frac{w'w}{d} \tag{28}$$

$$F_{T+1} = F_T + z'y - \frac{w'e_1}{d} \tag{29}$$

where

$$z = (y_1, x_1)$$

$$w = (e, 0)$$

$$e = y_1 - x(X'X)^{-1}X'Y_1$$

$$e_1 = y - x(X'X)^{-1}X'y_1(T)$$

$$d = 1 + x(X'X)^{-1}x'$$

Again using the matrix inversion lemma M2, we obtain from (28)

$$D_{T+1}^{-1} = (D_T + z'z)^{-1} + \frac{(D_T + z'z)^{-1}w'w(D_T + z'z)^{-1}}{d - w(D_T + z'z)^{-1}w'}$$

$$= D_T^{-1} - \frac{D_T^{-1}}{\alpha}[(d-k)z'z - nw'w + jw'z + jz'w]D_T^{-1} \tag{30}$$

where the following scalars are defined:

$$n = 1 + zD_T^{-1}z'$$

$$k = wD_T^{-1}w'$$

$$j = wD_T^{-1}z'$$

$$\alpha = n(d-k) + j^2.$$

Combining (29) and (30) and noting that

$$e_1 - w\hat{\delta}_1(T) = y - z\hat{\delta}_1(T)$$

gives the algorithm for updating the TSLS estimator:

$$\hat{\delta}_1(T+1) = \hat{\delta}_1(T) + D_T^{-1}(\alpha_1 z + \alpha_2 w)'[y - z\hat{\delta}_1(T)] \tag{31}$$

where

$$\alpha_1 = \frac{d - k + j}{\alpha}, \qquad \alpha_2 = \frac{j - n}{\alpha}.$$

The TSLS estimator can therefore be revised without any need for matrix inversion or storage of past data. The scalar $d$ is used to update estimates in each equation. The other scalars are easily calculated from the vectors $D_T^{-1}z'$ and $D_T^{-1}w'$. Calculation of the latter vector is simplified by noting that only $m_1$ elements are non-zero. In addition to $D_T^{-1}$, the matrix $(X'X)^{-1}X'Y_1$, which was calculated in obtaining $D_T^{-1}$, is required in storage.

## IV. Conclusions

In addition to providing a survey of recursive estimation algorithms which are likely to be of interest to economists, this paper gives a general algorithm useful for revising Aitken estimates as new data arrives. Formulae for updating estimators in two leading multiple equation models were also obtained.

*Queen's University,*
*Kingston, Canada*

## References

[1] Albert, A. *Regression and the Moore-Penrose Inverse*, New York: Academic Press, 1972.
[2] Albert, A. and R. Sittler. "A Method for Computing Least Squares Estimators that Keep up with the Data," *SIAM Journal of Control*, Vol. 3, 1965, pp. 394–417.
[3] American Statistical Association. "Regression Procedures for Missile Trajectory Estimation," *Proceedings of the 105th Regional Meeting*, Florida State University, 1965.
[4] Aoki, M. *Optimization of Stochastic Systems*, New York: Academic Press, 1967.
[5] Bartlett, M. S. "An Inverse Matrix Adjustment Arising in Discriminant Analysis," *Annals of Mathematical Statistics*, Vol. 22, 1951, pp. 107–111.
[6] Blum, M. "Fixed Memory Least Squares Filters Using Recursive Methods," *IRE Transactions on Information Theory*, Vol. 3, No. 3, Sept. 1957, pp. 178–182.
[7] ———. "A Stagewise Parameter Estimation Procedure for Correlated Data," *Numerishe Mathematik*, Vol. 3, 1961, pp. 202–208.
[8] ———. "Best Linear Unbiased Estimation by Recursive Methods," *SIAM Journal of Applied Mathematics*, Vol. 14. 1966, pp. 167–180.
[9] Boullion, T. and P. Odell. *Generalized Inverse Matrices*, New York: Wiley (Interscience), 1971.
[10] Chambers, J. M. "Regression Updating," *JASA*, Vol. 66, Dec. 1971, pp. 744–748.
[11] Duncan, D. B. and R. H. Jones. "Multiple Regression with Stationary Errors," *JASA*, Vol. 61, Dec. 1966, pp. 917–928.
[12] Fair, R. C. *A Short-Run Forecasting Model of the United States Economy*, Lexington, Mass.: D. C. Heath, 1971.
[13] Gauss, C. F. "Theoria Combinationis Observationum Erroribus Minimus Obnoxiac," *Werke*, Vol. 4, Gottingen, 1821 (collected works 1873).
[14] Goldfeld, S. M. "The Demand for Money Revisited," *Brookings Papers on Economic Activity*, Vol. 3, 1973, pp. 577–638.

[15] Golub, G. H. "Matrix Decomposition and Statistical Calculations," in R. C. Milton and J. A. Nelder (eds.), *Statistical Computation*. New York: Academic Press, 1969.

[16] Greville, T. N. E. "Some Applications of the Pseudo inverse of a Matrix," *SIAM Review*, Vol. 2, 1960, pp. 15–22.

[17] Householder, A. M. *The Theory of Matrices in Numerical Analysis*. New York: Blaisdell, 1964.

[18] Jones, R. H. "Recursive Estimation of a Subset of Regression Coefficients," *Annals of Mathematical Statistics*, Vol. 41, 1970, pp. 688–691.

[19] Lee, R. C. K. *Optimal Estimation, Identification and Control*. Research Monograph No. 28. Cambridge: MIT Press, 1964.

[20] Lewis, T. O and P. L. Odell. *Estimation in Linear Models*. Englewood Cliffs: Prentice-Hall, 1971.

[21] Odell, P. L. and T. O. Lewis. "Best Linear Recursive Estimation," *JASA*, Vol. 66, Dec. 1971, pp. 893–896.

[22] Plackett, R. L. "Some Theorems in Least Squares," *Biometrika*, Vol. 37, 1950, pp. 149–157.

[23] Sage, A. P. and J. L. Melsa. *Estimation Theory with Applications to Communication and Control*, New York: McGraw-Hill, 1971.

[24] Sherman, J. and W. J. Morrison. "Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix," *Annals of Mathematical Statistics*, Vol. 21, 1950, pp. 124–127.

[25] ——. "Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix," abstract, *Annals of Mathematical Statistics*, Vol. 20, 1949, p. 621.

[26] Tzafestas, S. G. "Some Computer-Aided Estimators in Stochastic Control Systems Identification," *International Journal of Control*, Vol. 12, 1970, pp. 385–399.